Journal of Statistics, Computing and Interdisciplinary Research
ISSN (P): 2707-7101

J. Stat. Comp. & Interdis. Rese.
Volume 2(2) (2020), 1-12
http://scir.wum.edu.pk/index.php/ojs

Using Auxiliary Information More Efficiently in Population Variance Estimation - A New Family of Estimators

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Abstract

In this article, we have suggested estimation of variance in finite population by using known values of parameter related to auxiliary information such as rank and second raw moment of auxiliary variable in stratified random sampling. The expression for the bias and mean squared error (MSE) of the suggested estimator are obtained up to first order of approximation. The proposed estimator is efficient comparatively various other estimators. A numerical and theoretical study are performed to support the suggested estimator.

Keywords: Bias, Mean squared error, Auxiliary variable, Ranks of auxiliary variable, Percentage relative efficiency. 1.5

Introduction

The etimation of finite population mean is always a laborious process due to limited respurces and sensitive nature of the study variables. In literature various studies are available to estimate the finete popuplation by the significant use of the auxiliary information.

Notations

Consider a finite population of size N denoted by $\delta = (\delta_1, \delta_2, \ldots, \delta_N)$. Further, let us assume that the population consist of L homogeneous partitions (starta), each of size N_h where $h = \{1,2, \ldots, L\}$, such that $\sum_{h=1}^{L} N_h = N$. For the purpose of consistency, we define Y, X, R and U be the study variable, auxiliary variable, ranks and squared values of auxiliary variable taking values Yih, Xih, Rih and Uih respectively, on the ith unit belongs to the hth stratum, where $i = \{1,2, \ldots, N_h\}$. Thus $W_h = N_h/N$ stays as the weight of hth stratum We then draw a sample of size nh from the hth stratum using SRS without replacement scheme for the estimation of population variance ensuring that the total sample size $n = \sum_{h=1}^{L} n_h$.

We now define population mean of study variable as $\bar{Y}_{st} = \bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$ where population mean of Y for hth stratum is $\bar{Y}_h = \sum_{i=1}^{N_h} Y_{ih}/N_h$. Similarly $\bar{X}_{st} = \bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ and $\bar{X}_h = \sum_{i=1}^{N_h} X_{ih}/N_h$ are the population mean of auxiliary variable and population mean of auxiliary in hth stratum, respectively. Furthermore, $\bar{R}_{st} = \bar{R} = \sum_{h=1}^{L} W_h \bar{R}_h$ and $\bar{R}_h = \sum_{i=1}^{N_h} R_{ih}/N_h$ represent the population mean of ranks and mean of ranks of hth stratum along with $\bar{U}_{st} = \bar{U} = \sum_{h=1}^{L} W_h \bar{U}_h$ and $\bar{U}_h = \sum_{i=1}^{N_h} U_{ih}/N_h$ define as population mean of squared values of auxiliary variable and population mean of squared values in hth stratum, respectively. Their corresponding sample estimate are given as $s_{y(h)}^2 = \hat{\bar{Y}} = \sum_{h=1}^{L} W_h \hat{\bar{Y}}_h$,

$\hat{\bar{Y}}_h = \sum_{i=1}^{n_h} Y_{ih}/n_h$, $\hat{\bar{X}}_{st} = \hat{\bar{X}} = \sum_{h=1}^{L} W_h \hat{\bar{X}}_h$, $\hat{\bar{X}}_h = \sum_{i=1}^{n_h} X_{ih}/n_h$, $\hat{\bar{R}}_{st} = \hat{\bar{R}} = \sum_{h=1}^{L} W_h \hat{\bar{R}}_h$, $\hat{\bar{R}}_h = \sum_{i=1}^{n_h} R_{ih}/n_h$, $\hat{\bar{U}}_{st} = \hat{\bar{U}} = \sum_{h=1}^{L} W_h \hat{\bar{U}}_h$ and $\hat{\bar{U}}_h = \sum_{i=1}^{n_h} U_{ih}/n_h$. Next, we define expression of population variances within stratum such that $S_{Y_h}^2 = \sum_{i=1}^{N_h} (Y_{ih} - \bar{Y}_h)^2/(N_h - 1)$, $S_{X_h}^2 = \sum_{i=1}^{N_h} (X_{ih} - \bar{X}_h)^2/(N_h - 1)$, $S_{R_h}^2 = \sum_{i=1}^{N_h} (R_{ih} - \bar{R}_h)^2/(N_h - 1)$ and $S_{U_h}^2 = \sum_{i=1}^{N_h} (U_{ih} - \bar{U}_h)^2/(N_h - 1)$, where as, covariances are given as $S_{Y_h X_h} = \sum_{i=1}^{N_h} (Y_{ih} - \bar{Y}_h)(X_{ih} - \bar{X}_h)/(N_h - 1)$, $S_{Y_h R_h} = \sum_{i=1}^{N_h} (Y_{ih} - \bar{Y}_h)(R_{ih} - \bar{R}_h)/(N_h - 1)$, $S_{Y_h U_h} = \sum_{i=1}^{N_h} (Y_{ih} - \bar{Y}_h)(U_{ih} - \bar{U}_h)/(N_h - 1)$, $S_{X_h R_h} = \sum_{i=1}^{N_h} (X_{ih} - \bar{X}_h)(R_{ih} - \bar{R}_h)/(N_h - 1)$,

$S_{X_h U_h} = \sum_{i=1}^{N_h} (X_{ih} - \bar{X}_h)(U_{ih} - \bar{U}_h)/(N_h - 1)$ and $S_{R_h U_h} = \sum_{i=1}^{N_h} (R_{ih} - \bar{R}_h)(U_{ih} - \bar{R}_h)/(N_h - 1)$.

Based on above provided expressions, we now provide correlation coefficients when stratified sampling scheme is used, such as

$$\rho_{YX(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_h X_h} S_{Y_h} S_{X_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2},$$

$$\rho_{YR(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_h R_h} S_{Y_h} S_{R_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{R_h}^2},$$

$$\rho_{YU(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_h U_h} S_{Y_h} S_{U_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{U_h}^2},$$

$$\rho_{XR(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{X_h R_h} S_{X_h} S_{R_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{R_h}^2},$$

$$\rho_{XU(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{X_h U_h} S_{X_h} S_{U_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{U_h}^2} \text{ and}$$

$$\rho_{RU(st)} = \sum_{h=1}^{L} W_h^2 \lambda \rho_{R_h U_h} S_{R_h} S_{U_h} / \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{U_h}^2},$$

where $\rho_{Y_h X_h} = S_{Y_h X_h}/S_{Y_h} S_{X_h}$,

$\rho_{Y_h R_h} = S_{Y_h R_h}/S_{Y_h} S_{R_h}$, $\rho_{Y_h U_h} = S_{Y_h U_h}/S_{Y_h} S_{U_h}$, $\rho_{X_h R_h} = S_{X_h R_h}/S_{X_h} S_{R_h}$, $\rho_{X_h U_h} = S_{X_h U_h}/S_{X_h} S_{U_h}$ and $\rho_{R_h U_h} = S_{R_h R_h}/S_{R_h} S_{U_h}$.

For further mathematical proceeding, the relative error terms are define as

$e_{0h} = (s_{y(h)}^2 - S_{Y(h)}^2)/S_{Y(h)}^2$, $e_{1h} = (s_{x(h)}^2 - S_{X(h)}^2)/S_{X(h)}^2$, $e_{2h} = (s_{r(h)}^2 - S_{R(h)}^2)/S_{R(h)}^2$, $e_{3h} = (s_{u(h)}^2 - S_{U(h)}^2)/S_{U(h)}^2$ and $\gamma_h = 1/n_h$.

Moreover, for $i = 0,1,2,3$, $E(e_{ih}) = 0$.

where as $E(e_{0h}^2) = \gamma_h \beta_{2y(h)}^*$, $E(e_{1h}^2) = \gamma_h \beta_{2x(h)}^*$, $E(e_{2h}^2) = \gamma_h \beta_{2r(h)}^*$, $E(e_{3h}^2) = \gamma_h \beta_{2u(h)}^*$, $E(e_{0h}e_{1h}) = \gamma_h \lambda_{2200(h)}^*$, $E(e_{0h}e_{2h}) = \gamma_h \lambda_{2020(h)}^*$, $E(e_{0h}e_{3h}) = \gamma_h \lambda_{2002(h)}^*$, $E(e_{1h}e_{2h}) = \gamma_h \lambda_{0220(h)}^*$, $E(e_{1h}e_{3h}) = \gamma_h \lambda_{0202(h)}^*$, $E(e_{2h}e_{3h}) = \gamma_h \lambda_{0022(h)}^*$.

Where $\beta_{2y(h)} = \frac{\mu_{4000(h)}}{\mu_{2000(h)}^2}$, $\beta_{2x(h)} = \frac{\mu_{0400(h)}}{\mu_{0200(h)}^2}$, $\beta_{2r(h)} = \frac{\mu_{0040(h)}}{\mu_{0020(h)}^2}$, $\beta_{2u(h)} = \frac{\mu_{0004(h)}}{\mu_{0002(h)}^2}$,

are the coefficient of kurtosis of y, x, r and u respectively.

Let $\beta_{2y(h)}^* = \beta_{2y(h)} - 1$, $\beta_{2x(h)}^* = \beta_{2x(h)} - 1$, $\beta_{2r(h)}^* = \beta_{2r(h)} - 1$, $\beta_{2u(h)}^* = \beta_{2u(h)} - 1$, $\lambda_{2200(h)}^* = \lambda_{2200(h)} - 1$, $\lambda_{2020(h)}^* = \lambda_{2020(h)} - 1$, $\lambda_{2002(h)}^* = \lambda_{2002(h)} - 1$, $\lambda_{0220(h)}^* = \lambda_{0220(h)} - 1$, $\lambda_{0202(h)}^* = \lambda_{0202(h)} - 1$, $\lambda_{0022(h)}^* = \lambda_{0022(h)} - 1$.

Where $\lambda_{pqrs(h)} = \frac{\mu_{pqrs(h)}}{\mu_{2000(h)}^{p/2}\mu_{0200(h)}^{q/2}\mu_{0020(h)}^{r/2}\mu_{0002(h)}^{s/2}}$, $\mu_{pqrs(h)} = \frac{1}{N_h}\sum_{i=1}^{N} (y_{hi} - \bar{Y})^p (x_{hi} - \bar{X})^q (r_{hi} - \bar{R})^r (u_{hi} - \bar{U})^s$.

$$\rho_{s_{y(h)}^2 s_{x(h)}^2} = \frac{\lambda_{2200(h)}^*}{\sqrt{\beta_{2y(h)}^*}\sqrt{\beta_{2x(h)}^*}}, \rho_{s_{y(h)}^2 s_{r(h)}^2} = \frac{\lambda_{2020(h)}^*}{\sqrt{\beta_{2y(h)}^*}\sqrt{\beta_{2r(h)}^*}}, \quad \rho_{s_{y(h)}^2 s_{u(h)}^2} = \frac{\lambda_{2002(h)}^*}{\sqrt{\beta_{2y(h)}^*}\sqrt{\beta_{2u(h)}^*}},$$

$$\rho_{s_{x(h)}^2 s_{r(h)}^2} = \frac{\lambda_{0220(h)}^*}{\sqrt{\beta_{2x(h)}^*}\sqrt{\beta_{2r(h)}^*}}, \quad \rho_{s_{x(h)}^2 s_{u(h)}^2} = \frac{\lambda_{0202(h)}^*}{\sqrt{\beta_{2x(h)}^*}\sqrt{\beta_{2u(h)}^*}}, \text{ and } \rho_{s_{r(h)}^2 s_{u(h)}^2} = \frac{\lambda_{0022(h)}^*}{\sqrt{\beta_{2r(h)}^*}\sqrt{\beta_{2u(h)}^*}}.$$

see Garcia and Cebrian (1996)

Existing Estimators of Variance in Stratified Random Sampling (VStRS)

We consider the existing estimators for the population variance $S_{Y(h)}^2$. Recall that the variance of the conventional unbiased variance estimator s_y^2(the sample variance), the variance is given by:

1.

$$\text{Var}(\hat{S}_y^2) = \sum_{h=1}^{L} W_h^4 \gamma_h^3 S_{y(h)}^4 \beta_{2y(h)}^*. \tag{1}$$

2. The Isaki (1983) introduced the following ratio estimator for population variance is given by:

$$\hat{S}_{R(st)}^2 = s_{y(h)}^2 \left(\frac{S_{x(h)}^2}{s_{x(h)}^2}\right), \tag{2}$$

The MSE of $\hat{S}_{R(st)}^2$ up to first degree of approximation, is defined as:

$$\text{MSE}(\hat{S}_{R(st)}^2) = \sum_{h=1}^{L} W_h^4 \gamma_h^3 S_{y(h)}^4 \left[\beta_{2y(h)}^* + \beta_{2x(h)}^* - 2\lambda_{2200(h)}^*\right]. \tag{3}$$

3. The traditional regression estimator defined by Isaki (1983) $\hat{S}_{Reg(st)}^2$ is given by

$$\hat{S}_{Reg(st)}^2 = s_{y(h)}^2 + b_h\left(S_{x(h)}^2 - s_{x(h)}^2\right), \tag{4}$$

Where b_h is the sample regression coefficient of y_h on x_h with corresponding population regression coefficient given by $\beta_h = \frac{s_{y(h)}^2 \lambda_{2200(h)}^*}{s_{x(h)}^2 \beta_{2x(h)}^*}$ in stratum h. The variance of $\hat{S}_{Reg(st)}^2$, to the first degree of approximation, is given by

$$\text{Var}(\hat{S}^2_{Reg(st)}) = \sum_{h=1}^{L} W_h^4 \gamma_h^3 S_{y(h)}^4 \beta_{2y(h)}^* \left(1 - \rho^2_{s^2_{y(h)} s^2_{x(h)}}\right). \quad (5)$$

4. Singh (1988) difference estimator is given by

$$\hat{S}^2_{D(st)} = w_1 s^2_{y(h)} + w_2 (S^2_{X(h)} - s^2_{x(h)}), \quad (6)$$

where w_1 and w_2 are suitable constants having optimum values are

$$w_{1(opt)} = \frac{1}{1+\gamma_h \beta_{2y(h)}^* \left(1-\rho^2_{s^2_{y(h)} s^2_{x(h)}}\right)} \text{ and } w_{2(opt)} = w_{1(opt)} \frac{S^2_{y(h)} \lambda^*_{2200(h)}}{S^2_{x(h)} \beta^*_{2x(h)}}.$$

The MSE of $\hat{S}^2_{D(st)}$, to the first degree of approximation is given by

$$MSE_{min}(\hat{S}^2_{D(st)}) \cong \sum_{h=1}^{L} W_h^4 \gamma_h^2 \left[\frac{\gamma_h S^4_{y(h)} \beta^*_{2y(h)} \left(1-\rho^2_{s^2_{y(h)} s^2_{x(h)}}\right)}{1+\gamma_h \beta^*_{2y(h)} \left(1-\rho^2_{s^2_{y(h)} s^2_{x(h)}}\right)}\right]. \quad (7)$$

5. The Bahl and Tuteja (1991) exponential ratio estimator is given by

$$\hat{S}^2_{BT,R(st)} = s^2_{y(h)} \exp\left(\frac{S^2_{X(h)} - s^2_{x(h)}}{S^2_{X(h)} + s^2_{x(h)}}\right). \quad (8)$$

The MSE of $\hat{S}^2_{BT,R(st)}$, up to first degree of approximation, is given by

$$\text{MSE}(\hat{S}_{BT,R(st)}) \cong \sum_{h=1}^{L} W_h^4 \gamma_h^3 S^4_{y(h)} \left(\beta^*_{2y(h)} + \frac{\beta^*_{2x(h)}}{4} - \lambda^*_{2200(h)}\right). \quad (9)$$

6. Shabbir and Gupta (2007) proposed the following estimator for $\hat{S}^2_{SG(st)}$ is given by

$$\hat{S}^2_{SG(st)} = (w_3 s^2_{y(h)} + w_4 (S^2_{X(h)} - s^2_{x(h)})) \exp\left(\frac{S^2_{X(h)} - s^2_{x(h)}}{S^2_{X(h)} + s^2_{x(h)}}\right).$$

where w_3 and w_4 are suitably chosen constants having optimum values given by

$$w_{3(opt)} = \frac{\gamma_h \beta^*_{2x(h)}}{8} \left[\frac{8-\gamma_h \beta^*_{2x(h)}}{\gamma_h (\beta^*_{2x(h)} + \beta^*_{2y(h)} \beta^*_{2x(h)} - \lambda^{*2}_{2200(h)})}\right]$$

and

$$w_{4(opt)} = \frac{S^2_{y(h)}}{8 S^2_{x(h)}} \left(\frac{-3\beta^*_{2x(h)} + 8\lambda^{*2}_{2200(h)} - \beta^*_{2x(h)} \lambda^{*2}_{2200(h)} + 4\beta^*_{2y(h)} \beta^*_{2x(h)} - 4\lambda^{*2}_{2200(h)}}{\beta^*_{2x(h)} + \beta^*_{2y(h)} \beta^*_{2x(h)} - \lambda^{*2}_{2200(h)}}\right)$$

The MSE of $\hat{S}^2_{SG(st)}$, up to first degree of approximation, is given by

$$\text{MSE}(\hat{S}^2_{SG(st)}) = \sum_{h=1}^{L} W_h^4 \gamma_h^2 \frac{S^4_{y(h)}}{64} \left[\frac{\gamma_h (-\beta^{*2}_{2x(h)} - 16\beta^*_{2y(h)} (1-\rho^2_{s^2_{y(h)} s^2_{x(h)}})(\beta^*_{2x(h)} - 4))}{1+\gamma_h \beta^*_{2y(h)} (1-\rho^2_{s^2_{y(h)} s^2_{x(h)}})}\right] \quad (10)$$

7. Singh (2014a) proposed an improved estimator for $\hat{S}^2_{SM(st)}$, is given by

$$\hat{S}^2_{SM(st)} = s^2_{y(h)} \left[w_5 + w_6 (S^2_{x(h)} - s^2_{x(h)})\right] \exp\left\{\Psi \frac{a(S^2_{x(h)} - s^2_{x(h)})}{a(S^2_{x(h)} + s^2_{x(h)}) + 2b}\right\}, \quad (11)$$

where w_5 and w_6 are suitably constants, Ψ takes values $+1$ and -1 for ratio and product estimators and a, b be the known population parameters of the auxiliary variables. The minimum MSE of $\hat{S}^2_{SM(st)}$, up to first degree of approximation, at optimum values $g = \Psi = 1$,

$$w_{5(opt)} = \frac{1}{4} \left[\frac{-12\beta^*_{2x(h)} \lambda^*_{2200(h)} + 3\beta^{*2}_{2x(h)} + 16\lambda^{*2}_{2200(h)} - 8\beta^*_{2x(h)} - 2\beta^{*2}_{2x(h)}}{\beta^{2*}_{2x(h)} - 4\beta^*_{2x(h)} \lambda^*_{2200(h)} + 8\lambda^{*2}_{2200(h)} - 2\beta^*_{2y(h)} \beta^*_{2x(h)} - \beta^*_{2x(h)} - \beta^{*2}_{2x(h)}}\right]$$

and

$$w_{6(opt)} = \frac{1}{4S^2_{x(h)}}\left[\frac{-6\beta^*_{2x(h)}\lambda^*_{2200(h)} + \beta^{*2}_{2x(h)} + 8\lambda^*_{2200(h)} - 4\beta^*_{2x(h)} + 8\lambda^*_{2200(h)} - 8\beta^*_{2x(h)}\lambda^*_{2200(h)} + 4\beta^*_{2x(h)}\beta^*_{2y(h)}}{\beta^{2*}_{2x(h)} - 4\beta^*_{2x(h)}\lambda^*_{2200(h)} + 8\lambda^{*2}_{2200(h)} - 2\beta^*_{2y(h)}\beta^*_{2x(h)} - \beta^*_{2x(h)} - \beta^{*2}_{2x(h)}}\right]$$

is given

$$\text{MSE}(\hat{S}^2_{SM(st)})_{min} \cong \sum_{h=1}^{L} W^4_h\gamma^3_h \frac{S^2_{y(h)}}{64}\left[\frac{\beta^*_{2x(h)}\left[\begin{array}{c}\beta^*_{2x(h)}(\beta^*_{2x(h)}+8\lambda^*_{2200(h)})+16(\beta^*_{2x(h)} \\ -4\text{Var}(\hat{S}^2_{Reg(st)}))+16\lambda^*_{2200(h)}(\lambda^*_{2200(h)}\beta^*_{2x(h)})\end{array}\right]}{-\beta^*_{2x(h)}(1+\beta^*_{2y(h)}+2\lambda^*_{2200(h)}))+4\lambda^{*2}_{2200(h)})}\right] \quad (12)$$

8. Yaqub and Shabbir (2016) proposed an improved class of estimators for $\hat{S}^2_{YS(st)}$, is given by

$$\hat{S}^2_{YS(st)} = s^2_{y(h)}\left[w_7 + w_8(S^2_{x(h)} - s^2_{x(h)})\right]\left(\frac{a(S^2_{x(h)}-s^2_{x(h)})}{a(S^2_{x(h)}+s^2_{x(h)})+2b}\right)\left[\begin{array}{c}\frac{1}{2}\exp\left\{\frac{a(S^2_{x(h)}-s^2_{x(h)})}{a(S^2_{x(h)}+s^2_{x(h)})+2b}\right\} \\ +\frac{1}{2}\exp\left\{\frac{a(S^2_{x(h)}-s^2_{x(h)})}{a(S^2_{x(h)}+s^2_{x(h)})+2b}\right\}\end{array}\right], \quad (13)$$

where w_7 and w_{10} are suitably constants and a and b be the known population parameters of the auxiliary variables. The minimum MSE of $\hat{S}^2_{YS(st)}$, up to first degree of approximation, at optimum values $a = 1, b = 0$,

$$w_{9(opt)} = \frac{\beta^*_{2x(h)}}{2}\left[\frac{1+7(1-\beta^*_{2x(h)})}{\beta^{*2}_{2x(h)}+4\beta^*_{2x(h)}(1-\beta^*_{2x(h)})+4\beta^*_{2y(h)}\beta^*_{2x(h)}-4\lambda^*_{2200(h)}}\right]$$

and

$$w_{10(opt)} = \frac{S^2_{y(h)}}{2S^2_{x(h)}}\left[\frac{\lambda^*_{2200(h)} + 7\lambda^*_{2200(h)}(1-\beta^*_{2x(h)}) - 8\beta^*_{2x(h)}(1-\beta^*_{2x(h)}) + 8\beta^*_{2y(h)}\beta^*_{2x(h)} - 8\lambda^{*2}_{2200(h)}}{\beta^{*2}_{2x(h)} + 4\beta^*_{2x(h)}(1-\beta^*_{2x(h)}) + 4\beta^*_{2y(h)}\beta^*_{2x(h)} - 4\lambda^*_{2200(h)}}\right],$$

is given by

$$\text{MSE}(\hat{S}^2_{YS(st)})_{min} \cong \sum_{h=1}^{L} W^4_h\gamma^3_h \frac{S^4_{y(h)}}{16}\left[\frac{64S^{-4}_{y(h)}\text{Var}(\hat{S}^2_{Reg(st)})(1-\beta^*_{2x(h)})-\beta^{*2}_{2x(h)}}{\beta^*_{2x(h)}+4(1-\beta^*_{2x(h)})+4S^{-4}_{y(h)}\text{Var}(\hat{S}^2_{Reg(st)})}\right]. \quad (14)$$

9. Muneer (2018) proposed an ratio-product type exponential estimator for $\hat{S}^2_{M(st)}$, is given by

$$\hat{S}^2_{M(st)} = S^2_{y(h)}\left[w_{11}\left(\frac{S^2_{x(h)}}{s^2_{x(h)}}\right) + w_{12}\left(\frac{s^2_{x(h)}}{S^2_{x(h)}}\right)\right]\exp\left(\frac{S^2_{x(h)}-s^2_{x(h)}}{S^2_{x(h)}+s^2_{x(h)}}\right), \quad (15)$$

where w_{11} and w_{12} are unknown constants whose values are

$$w_{11(opt)} = \frac{1}{8}\left[\frac{16\lambda^{*2}_{2200(h)} + 16\lambda^*_{2200(h)}\beta^*_{2y(h)} - 24\lambda^*_{2200(h)}\beta^*_{2x(h)} - 16\beta^*_{2y(h)}\beta^*_{2x(h)} - \beta^{*2}_{2x(h)} - 16\lambda^*_{2200(h)} - 8\beta^*_{2x(h)}}{16\lambda^{*2}_{2200(h)} - 16\lambda^*_{2200(h)}\beta^*_{2x(h)} - 4\beta^*_{2y(h)}\beta^*_{2x(h)} + \beta^{*2}_{2x(h)} - 4\beta^*_{2x(h)}}\right]$$

and

$$w_{12(opt)} = \frac{S_{y(h)}^4}{16}\left\{\frac{\left[\begin{array}{c} 48\lambda_{2200(h)}^{*2} - 16\lambda_{2200(h)}^{*}\beta_{2y(h)}^{*} - 72\lambda_{2200(h)}^{*}\beta_{2x(h)}^{*} + 16\beta_{2y(h)}^{*}\beta_{2x(h)}^{*} + 21\beta_{2x(h)}^{*2} \\ +16\lambda_{2200(h)}^{*} - 24\beta_{2x(h)}^{*} \end{array}\right]}{16\lambda_{2200(h)}^{*2} - 16\lambda_{2200(h)}^{*}\beta_{2x(h)}^{*} - 4\beta_{2y(h)}^{*}\beta_{2x(h)}^{*} + \beta_{2x(h)}^{*2} - 4\beta_{2x(h)}^{*}}\right\}.$$

The minimum MSE of $\hat{S}_{YS(st)}^2$, up to first degree of approximation, is given by

$$\text{MSE}_{min}(\hat{S}^2 M(st)) \cong \sum_{h=1}^{L} W_h^4 \gamma_h^3 \frac{S_{y(h)}^4}{16}\left\{\frac{\left[\begin{array}{c} 64\lambda_{2200(h)}^{*2}\beta_{2y(h)}^{*} - 48\lambda_{2200(h)}^{*}\beta_{2x(h)}^{*} - 128\lambda_{2200(h)}^{*}\beta_{2y(h)}^{*}\beta_{2x(h)}^{*} \\ +48\lambda_{2200(h)}^{*}\beta_{2x(h)}^{*2}\beta_{2x(h)}^{*} + 64\beta_{2y(h)}^{*}\beta_{2x(h)}^{*2} + 9\beta_{2x(h)}^{*3} \\ +64\lambda_{2200(h)}^{*2} - 64\beta_{2y(h)}^{*}\beta_{2x(h)}^{*} \end{array}\right]}{16\lambda_{2200(h)}^{*2} - 16\lambda_{2200(h)}^{*}\beta_{2x(h)}^{*} - 4\beta_{2y(h)}^{*}\beta_{2x(h)}^{*} + \beta_{2x(h)}^{*2} - 4\beta_{2x(h)}^{*}}\right\}. \quad (16)$$

Proposed Estimator

We propose a difference ratio type estimator of the population variance $\hat{S}_y^2$ in stratified random sampling as

$$\hat{S}_{Pr(st)}^2 = \{w_{13}s_{y(h)}^2 + w_{14}(S_{x(h)}^2 - s_{x(h)}^2) + w_{15}(S_{r(h)}^2 - s_{r(h)}^2) + w_{16}(S_{u(h)}^2 - s_{u(h)}^2)\} \exp\left(\frac{a(S_{x(h)}^2 - s_{x(h)}^2)}{a(S_{x(h)}^2 + s_{x(h)}^2) + 2b}\right), \quad (17)$$

where w_{13}, w_{14}, w_{15} and w_{16} are unknown constants, whose values to be determined for optimality. After rewriting the proposed estimator $\hat{S}_{Pr(st)}^2$ in terms of relative error. Moreover, a and b can take varying values and thus provide different members of our proposed family of estimators. Table 1 below, presents various values of a and b and resultant estimators.

$$\hat{S}_{Pr(st)}^2 = \{w_{13}S_{y(h)}^2 + S_{y(h)}^2 e_{0h} w_{13} - S_{x(h)}^2 e_{1h} w_{14} - S_r^2 e_{2h} w_{15} - S_{u(h)}^2 e_{3h} w_{16}\} \{1 - \frac{\theta e_{1h}}{2} + \frac{3\theta^2 e_{1h}^2}{8} + \cdots\}. \quad (18)$$

solving further the right hand side of (23), and keeping terms only up to second order in $e_{ih}s$, we can write

$$\left(\hat{S}_{Pr(st)}^2 - S_{y(h)}^2\right) = S_{y(h)}^2(w_{13} - 1) + S_{y(h)}^2 e_{0h} w_{13} - S_{x(h)}^2 e_{1h} w_{14} - S_{r(h)}^2 e_{2h} w_{15} - S_{u(h)}^2 e_{3h} w_{16} - \frac{1}{2}\theta S_{y(h)}^2 e_{1h} w_{13} + \frac{3}{8}\theta^2 S_{y(h)}^2 e_{1h}^2 w_{13} - \frac{1}{2}\theta S_{y(h)}^2 e_{0h} e_1 w_{13} + \frac{1}{2}\theta S_{x(h)}^2 e_{1h}^2 w_{14} + \frac{1}{2}\theta S_{r(h)}^2 e_1 e_{2h} w_{15} + \frac{1}{2}\theta S_{u(h)}^2 e_{1h} e_{3h} w_{16}. \quad (19)$$

By applying the expectation on both sides on (24), we obtain the bias of an estimator $\hat{S}_{Pr(st)}^2$ as

$$\text{Bias}(\hat{S}_{Pr(st)}^2) = \sum_{h=1}^{L} W_h^2 \left[\begin{array}{c} S_{y(h)}^2(w_{13} - 1) + \frac{3}{8}\gamma_h^2\theta^2 S_{y(h)}^2 \beta_{2x(h)}^{*} w_{13} - \frac{1}{2}\gamma_h^2\theta S_{y(h)}^2 w_{13}\lambda_{2200(h)}^{*} \\ +\frac{1}{2}\gamma_h^2\theta S_{x(h)}^2 \beta_{2x(h)}^{*} w_{14} + \frac{1}{2}\gamma_h^2\theta S_{r(h)}^2 w_{15}\lambda_{0220(h)}^{*} + \frac{1}{2}\gamma_h^2\theta S_{u(h)}^2 w_{16}\lambda_{0202(h)}^{*} \end{array}\right]. \quad (20)$$

Squaring and then taking expectation of both sides of (24) we get the MSE of an estimator

$\hat{S}^2_{Pr(st)}$, up to first degree of approximation, as

$$\mathrm{MSE}(\hat{S}^2_{Pr(st)}) \cong \sum_{h=1}^{L} W_h^4 \left[\begin{array}{l} S^4_{y(h)}(1-2w_{13}) + S^4_{y(h)}w_{13}^2 + \gamma_h^4\theta^2 S^4_{y(h)}\beta^*_{2x(h)}w_{13}^2 - \frac{3}{4}\gamma_h^4\theta^2 S^4_{y(h)}\beta^*_{2x(h)}w_{13} \\ +\gamma_h^4 S^4_{y(h)}\beta^*_{2y(h)}w_{13}^2 + \gamma_h^4 S^4_{x(h)}\beta^*_{2x(h)}w_{14}^2 + \gamma_h^4 S^4_{r(h)}\beta^*_{2r(h)}w_{15}^2 \\ +\gamma_h^4 S^4_{u(h)}\beta^*_{2u(h)}w_{16}^2 + 2\gamma_h^4 S^2_{x(h)}S^2_{r(h)}w_{14}w_{15}\lambda^*_{0220(h)} + 2\gamma_h^4 S^2_{x(h)}S^2_{u(h)} \\ w_{14}w_{16}\lambda^*_{0202(h)} + 2\gamma_h^4 S^2_{r(h)}S^2_{u(h)}w_{15}w_{16}\lambda^*_{0022(h)} + 2\gamma_h^4\theta S^2_{y(h)}S^2_{r(h)} \\ w_{13}w_{15}\lambda^*_{0220(h)} + 2\gamma_h^4\theta S^2_{y(h)}S^2_{u(h)}w_{13}w_{16}\lambda^*_{0220(h)} - 2\gamma_h^4 S^2_{y(h)}S^2_{x(h)}w_{13}w_{14} \\ \lambda^*_{2200(h)} - 2\gamma_h^4 S^2_{y(h)}S^2_{r(h)}w_{13}w_{15}\lambda^*_{2002(h)} - 2\gamma_h^4 S^2_{y(h)}S^2_{u(h)}w_{13}w_{16}\lambda^*_{2020(h)} \\ -\gamma_h^4\theta S^2_{y(h)}S^2_{r(h)}w_{15}\lambda^*_{0220(h)} - \gamma_h^4\theta S^2_{x(h)}S^2_{u(h)}w_{16}\lambda^*_{0202(h)} \\ +2\gamma_h^4\theta S^2_{y(h)}S^2_{x(h)}\beta^*_{2x(h)}w_{13}w_{14} + \gamma_h^4\theta S^4_{y(h)}w_{13}\lambda^*_{2200(h)} - \gamma_h^4\theta S^2_{y(h)}S^2_{x(h)} \\ \beta^*_{2x(h)}w_{14} - 2\gamma_h^4\theta S^4_{y(h)}w_{13}^2\lambda^*_{2200(h)} \end{array} \right]$$

By minimizing (26) we have the optimum values of w_{13}, w_{14}, w_{15} and w_{16}, respectively, given by

$$w_{13(opt)} = \frac{(B_6\beta^*_{2x(h)}+A_2)(\gamma_h\theta^2\beta^*_{2x(h)}-8)}{-8\left[\begin{array}{l} (-A_4\lambda^{*2}_{0202(h)}-2(A_5+A_6)+A_7-2A_8+A_9\lambda^{*2}_{0022(h)} \\ +2\beta^*_{2x(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)}+(A_9\beta^*_{2u(h)}+\beta^*_{2x(h)}\beta^*_{2r(h)}\lambda^{*2}_{2002(h)}) \\ +\beta^*_{2u(h)}\beta^*_{2x(h)}\lambda^{*2}_{2020(h)})\gamma_h+A_2+B_6\beta^*_{2x(h)} \end{array}\right]},$$

$$w_{14(opt)} = \frac{S^2_{y(h)}\left[\begin{array}{l} \frac{-1}{4}\gamma_h\theta^3\beta^{*2}_{2x(h)}+A_2\theta^3\beta^*_{2x(h)}-\frac{1}{4}B_1\gamma_h\theta^2\beta^*_{2x(h)}+((B_2+B_3\beta^*_{2r(h)})\beta^*_{2x(h)} \\ +\lambda^{*2}_{2200(h)}\lambda^{*2}_{0022(h)}+A_9\lambda^*_{0022(h)}+A_4\lambda^{*2}_{0202(h)}-2B_5\lambda^*_{2002(h)}\lambda^*_{0202(h)}+A_7 \\ +A_{10}+B_6\beta^*_{2x(h)}+A_2)\theta+2B_1 \end{array}\right]}{2S^2_{x(h)}\left[\begin{array}{l} (-A_4\lambda^{*2}_{0202(h)}-2(A_5+A_6)+A_7-2A_8+A_9\lambda^{*2}_{0022(h)} \\ +2\beta^*_{2x(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)}+(A_9\beta^*_{2u(h)}+\beta^*_{2x(h)}\lambda^{*2}_{2002(h)})\beta^*_{2r(h)} \\ +\beta^*_{2u(h)}\beta^*_{2x(h)}\lambda^{*2}_{2020(h)})\gamma_h+A_2+B_6\beta^*_{2x(h)} \end{array}\right]},$$

$$w_{15(opt)} = \frac{S^2_{y(h)}(B_7+B_8)(\gamma_h\theta^2\beta^*_{2x(h)}-8)}{-8S^2_{r(h)}\left[\begin{array}{l} (-A_4\lambda^{*2}_{0202(h)}-2(A_5+A_6)+A_7-2A_8+A_9\lambda^{*2}_{0022(h)} \\ +2\beta^*_{2x(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)}+(A_9\beta^*_{2u(h)}+\beta^*_{2x(h)}\lambda^{*2}_{2002(h)})\beta^*_{2r(h)} \\ +\lambda^{*2}_{2020(h)}\beta^*_{2u(h)}\beta^*_{2x(h)})\gamma_h+A_2+B_6\beta^*_{2x(h)} \end{array}\right]},$$

and

$$w_{16(opt)} = \frac{S^2_{y(h)}(B_9+B_{10})(\gamma_h\theta^2\beta^*_{2x(h)}-8)}{-8S^2_{u(h)}\left[\begin{array}{l} (-A_4\lambda^{*2}_{0202(h)}-2(A_5+A_6)+A_7-2A_8+A_9\lambda^{*2}_{0022(h)} \\ +2\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)}\beta^*_{2x(h)}+(A_9\beta^*_{2u(h)}+\beta^*_{2x(h)}\lambda^{*2}_{2002(h)})\beta^*_{2r(h)} \\ +\beta^*_{2u(h)}\beta^*_{2x(h)}\lambda^{*2}_{2020(h)})\gamma_h+A_2+B_6\beta^*_{2x(h)} \end{array}\right]}.$$

Where $B_1 = (-\lambda^{*2}_{0022(h)}\lambda^*_{2200(h)} + (\lambda^*_{0202(h)}\lambda^*_{2020(h)})\lambda^*_{0022(h)} - \beta^*_{2r(h)}\lambda^*_{0202(h)}\lambda^*_{2002(h)}$
$-\beta^*_{2u(h)}(-\beta^*_{2r(h)}\lambda^*_{2200(h)} + \lambda^*_{0220(h)}\lambda^*_{2020(h)}))$,

$$B_2 = \lambda^{*2}_{0022(h)}\beta^*_{2y(h)} + \lambda^{*2}_{2020(h)}\beta^*_{2u(h)} - 2\lambda^*_{2002(h)}\lambda^*_{2020(h)}\lambda^*_{0022(h)},$$
$$B_3 = \beta^*_{2u(h)}\beta^*_{2y(h)} - \lambda^{*2}_{2002(h)}, B_4 = \beta^*_{2y(h)}\lambda^*_{0220(h)} - \lambda^*_{2020(h)}\lambda^*_{2200(h)},$$
$$B_5 = -\beta^*_{2r(h)}\lambda^*_{2200(h)} + \lambda^*_{0220(h)}\lambda^*_{2020(h)}, B_6 = (-\beta^*_{2r(h)}\beta^*_{2u(h)} + \lambda^{*2}_{0022(h)}),$$
$$B_7 = (\beta^*_{2u(h)}\lambda^*_{2020(h)} - \lambda^*_{0022(h)}\lambda^*_{2002(h)})\beta^*_{2x(h)},$$
$$B_8 = \lambda^{*2}_{0202(h)}\lambda^*_{2020(h)} + \beta^*_{2u(h)}\lambda^*_{0220(h)}\lambda^*_{2200(h)} - (\lambda^*_{0022(h)}\lambda^*_{2200(h)} + \lambda^*_{2002(h)})\lambda^*_{0202(h)},$$
$$B_9 = (-\beta^*_{2r(h)}\lambda^*_{2002(h)} + \lambda^*_{0022(h)}\lambda^*_{2020(h)})\beta^*_{2x(h)},$$
$$B_{10} = \lambda^{*2}_{0220(h)}\lambda^*_{2002(h)} + \beta^*_{2r(h)}\lambda^*_{0202(h)}\lambda^*_{2200(h)} - (\lambda^*_{0022(h)}\lambda^*_{2200(h)} + \lambda^*_{0202(h)}\lambda^*_{2020(h)})\lambda^*_{0220(h)},$$
$$A_1 = \beta^*_{2r(h)}\lambda^{*2}_{0202(h)} + \beta^*_{2u(h)}\lambda^{*2}_{0220(h)} - 2\lambda^*_{0022(h)}\lambda^*_{0202(h)}\lambda^*_{0220(h)},$$
$$A_2 = \lambda^{*2}_{0022(h)}\beta^*_{2y(h)} + \beta^*_{2r(h)}\lambda^{*2}_{2002(h)} - 2\lambda^*_{0022(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)},$$
$$A_3 = (-\beta^*_{2r(h)}\beta^*_{2y(h)} + \lambda^{*2}_{2020(h)}),$$
$$A_4 = (\lambda^*_{0022(h)}\beta^*_{2y(h)} - \lambda^*_{2002(h)}\lambda^*_{2020(h)})\lambda^*_{0220(h)}\lambda^*_{0202(h)},$$
$$A_5 = (\beta^*_{2r(h)}\lambda^*_{2002(h)} - \lambda^*_{0022(h)}\lambda^*_{2020(h)})\lambda^*_{2200(h)}\lambda^*_{0202(h)},$$
$$A_6 = (-\beta^*_{2u(h)}\beta^*_{2y(h)} + \lambda^{*2}_{2002(h)})\lambda^{*2}_{0220(h)},$$
$$A_7 = (-\beta^*_{2u(h)}\lambda^*_{2020(h)} + \lambda^*_{0022(h)}\lambda^*_{2002(h)})\lambda^*_{2200(h)}\lambda^*_{0220(h)},$$
$$A_8 = 2(B_4\lambda^*_{0202(h)} - \lambda^*_{0220(h)}\lambda^*_{2002(h)}\lambda^*_{2200(h)}),$$
$$A_9 = 2\lambda^*_{2020(h)}\lambda^*_{2200(h)}\lambda^*_{0220(h)}\beta^*_{2u(h)} - \lambda^{*2}_{2200(h)}\beta^*_{2r(h)}\beta^*_{2u(h)},$$
$$G = (A_4\lambda^{*2}_{0202(h)} + 2(A_5 + A_6) + A_7 - 2A_8).$$

Substitute the optimum vales of w_{13}, w_{14}, w_{15} and w_{16}, in (26), we get the minimum MSE of $\hat{S}^2_{Pr(st)}$, is given by

$$\text{MSE}_{min}(\hat{S}^2_{Pr(st)}) \cong \sum_{h=1}^{L} W_h^4\gamma_h^3 \frac{S^4_{y(h)}\left[\begin{array}{l}\frac{-1}{16}B_6\gamma_h\theta^4\beta^{*3}_{2x(h)} - \gamma_h\theta^2\beta^{*2}_{2x(h)}\{\frac{1}{16}A_2\theta^2 + A_3 \\ +A_4\beta^*_{2u(h)}\} + \{\gamma_h\theta^2\beta^*_{2x(h)} + G + B_6\lambda^{*2}_{2200(h)} \\ +4(A_3 + A_4\beta^*_{2u(h)})\} - 4(G + B_6\lambda^{*2}_{2200(h)})\end{array}\right]}{4\left[\begin{array}{l}\{-G - A_9\lambda^{*2}_{0022(h)} - 2\beta^*_{2x(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)} \\ +(A_9\beta^*_{2u(h)} + \beta^*_{2x(h)}\lambda^{*2}_{2002(h)})\beta^*_{2r(h)} \\ +\beta^*_{2u(h)}\beta^*_{2x(h)}\lambda^{*2}_{2020(h)}\}\gamma_h + A_2 + B_6\beta^*_{2x(h)}\end{array}\right]}. \quad (21)$$

[hp!]

Members of the suggested families of estimators

a	b	$\hat{S}^2_{Pr(st)}$	
1	$C_{x(st)}$	$\hat{S}^{2(1)}_{Pr(st)}$	$(w_{13}s^2_{y(h)} + w_{14}(S^2_{x(h)} - s^2_{x(h)}) + w_{15}(S^2_{r(h)} - s^2_{r(h)}) + w_{16}(S^2_{u(h)} - s^2_{u(h)}))\exp\left(\frac{(S^2_{x(h)} - s^2_{x(h)})}{(S^2_{x(h)} + s^2_{x(h)}) + 2C_{x(st)}}\right)$
	$\beta_{2(st)(x)}$	$\hat{S}^{2(2)}_{Pr(st)}$	$(w_{13}s^2_{y(h)} + w_{14}(S^2_{x(h)} - s^2_{x(h)}) + w_{15}(S^2_{r(h)} - s^2_{r(h)}) + w_{16}(S^2_{u(h)} -$

			$s^2_{u(h)}))\exp\left(\frac{(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\beta_{2(st)(x)}}\right)$
$\beta_{2(st)(x)}$	$C_{x(st)}$	$\hat{S}^{2(3)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{\beta_{2(st)(x)}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2C_{x(st)}}\right)$
$C_{x(st)}$	$\beta_{2(st)(x)}$	$\hat{S}^{2(4)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{C_{x(st)}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\beta_{2(st)(x)}}\right)$
	$\rho_{YX(st)}$	$\hat{S}^{2(5)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\rho_{YX(st)}}\right)$
$C_{x(st)}$	$\rho_{YX(st)}$	$\hat{S}^{2(6)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{C_{x(st)}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\rho_{YX(st)}}\right)$
$\rho_{YX(st)}$	$C_{x(st)}$	$\hat{S}^{2(7)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{\rho_{YX(st)}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2C_{x(st)}}\right)$
$\beta_{2(st)(x)}$	$\rho_{YX(st)}$	$\hat{S}^{2(8)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{\beta_{2(st)(x)}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\rho_{yx}}\right)$
$\rho_{YX(st)}$	$\beta_{2(st)(x)}$	$\hat{S}^{2(9)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{\rho_{yx}(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2\beta_{2(st)(x)}}\right)$
	$\mathrm{N}S^2_x$	$\hat{S}^{2(10)}_{Pr(st)}$	$(w_{13}s^2_{y(h)}+w_{14}(S^2_{x(h)}-s^2_{x(h)})+w_{15}(S^2_{r(h)}-s^2_{r(h)})+w_{16}(S^2_{u(h)}-s^2_{u(h)}))\exp\left(\frac{(S^2_{x(h)}-s^2_{x(h)})}{(S^2_{x(h)}+s^2_{x(h)})+2NS^2_x}\right)$

Performance Comparison

In this section, we advance by comparing the efficiency of our proposed family with Muneer et al.

(2018), we need to show $\mathrm{MSE}_{min}(\hat{S}^2_{M(st)}) - \mathrm{MSE}_{min}(\hat{S}^2_{Pr(st)}) > 0$, which on simplification provides the general efficiency condition such as,

$$\begin{bmatrix} 64\lambda^{*2}_{2200(h)}\beta^*_{2y(h)} - 48\lambda^*_{2200(h)}\beta^*_{2x(h)} - 128\lambda^*_{2200(h)}\beta^*_{2y(h)}\beta^*_{2x(h)} \\ +48\lambda^*_{2200(h)}\beta^{*2}_{2x(h)}\beta^*_{2x(h)} + 64\beta^*_{2y(h)}\beta^{*2}_{2x(h)} + 9\beta^{*3}_{2x(h)} \\ +64\lambda^{*2}_{2200(h)} - 64\beta^*_{2y(h)}\beta^*_{2x(h)} \end{bmatrix}$$
$$\begin{bmatrix} \{-G - A_9\lambda^{*2}_{0022(h)} - 2\beta^*_{2x(h)}\lambda^*_{2020(h)}\lambda^*_{2002(h)}\lambda^*_{0022(h)} \\ +(A_9\beta^*_{2u(h)} + \beta^*_{2x(h)}\lambda^{*2}_{2002(h)})\beta^*_{2r(h)} \\ +\beta^*_{2u(h)}\beta^*_{2x(h)}\lambda^{*2}_{2020(h)}\}\gamma_h + A_2 + A_1\beta^*_{2x(h)} \end{bmatrix}$$
$$-4\left[16\lambda^{*2}_{2200(h)} - 16\lambda^*_{2200(h)}\beta^*_{2x(h)} - 4\beta^*_{2y(h)}\beta^*_{2x(h)} + \beta^{*2}_{2x(h)} - 4\beta^*_{2x(h)}\right]$$
$$\begin{bmatrix} \frac{-1}{16}A_1\gamma_h\theta^4\beta^{*3}_{2x(h)} - \gamma_h\theta^2\beta^{*2}_{2x(h)}\{\frac{1}{16}A_2\theta^2 + A_3 \\ +A_4\beta^*_{2u(h)}\} + \{\gamma_h\theta^2\beta^*_{2x(h)} + G + A_1\lambda^{*2}_{2200(h)} \\ +4(A_3 + A_4\beta^*_{2u(h)})\} - 4(G + A_1\lambda^{*2}_{2200(h)}) \end{bmatrix}.$$

Empirical Study

For numerical comparisons of the proposed and existing estimators, a dataset is considered. The newly proposed estimators are compared in terms of $PREs$ and the results are reported in Table 1. The detail of the data set is given below.

Dataset: Source: Murthy (1967)

y: Output of the factory and x: number of workers.

N=80, n=10, $\bar{Y} = 5182.64$, $\bar{X} = 285.125$, $\bar{R} = 40.5$, $\bar{U} = 153514.2$, $S_y = 1835.659$, $S_x = 270.4294$, $S_r = 23.23749$, $S_u = 256931.1$,

$\rho_{yx} = 0.5334603$, $\rho_{yr} = 0.8305842$, $\rho_{yu} = 0.4397924$, $\rho_{xr} = 0.4284402$, $\rho_{xu} = 0.9329431$, $\rho_{ru} = 0.285485$, $C_y = 0.354194$, $C_x = 0.948459$, $C_r = 0.573765$, $C_u = 1.673663$, $\beta_{2(y)} = 2.238321$, $\beta_{2(x)} = 3.536017$, $\beta_{2(r)} = 1.776874$, $\beta_{2(u)} = 7.196021$, $\lambda_{2200} = 2.294326$, $\lambda_{2020} = 1.893889$, $\lambda_{2002} = 2.836948$, $\lambda_{0220} = 1.918748$, $\lambda_{0202} = 4.828777$, $\lambda_{0022} = 2.172258$, $\gamma = 0.1$.

$PREs = Var(\hat{S}^2_y)/MSE(\hat{S}^2_i) \times 100$; $i = y, R, Reg, RD, GK, AH, Pr$.

Table 1: The $PREs$ of estimators for different choices of a and b

S.No	a	b	$\widehat{S}^2_{GK}$	$\widehat{S}^2_{AH}$	$\widehat{S}^2_{Pr}$
1	1	C_x	212.5887	419.8847	992.6066
	1	$\beta_{2(x)}$	212.5869	419.8807	992.5903
	$\beta_{2(x)}$	C_x			

			212.5891	419.8858	992.6109
	C_x	$\beta_{2(x)}$	212.5867	419.8804	992.5891
	1	ρ_{yx}	212.5877	419.8826	992.5981
	C_x	ρ_{yx}	212.5876	419.8824	992.5973
	ρ_{yx}	C_x	212.5895	419.8855	992.6099
	$\beta_{2(x)}$	ρ_{yx}	212.5889	419.8852	992.6085
	ρ_{yx}	$\beta_{2(x)}$	212.5883	419.8838	992.6028
	1	NS_x^2	192.4908	380.7184	861.3078
$\hat{S}_y^2$			100.0000		
$\hat{S}_R^2$			104.4392		
$\hat{S}_{Reg}^2$			173.0900		
$\hat{S}_{R,D}^2$			188.4244		

From the results contained in table 1, it is much clear that the $PREs$ of the proposed estimators are significantly greater than those with their competitors, which shows the appropriateness of the new estimators.

Conclusion

On the basis of theoretical as well as numerical results, it is inferred that the suggested estimator is more efficient as compared to the existing mean, product, classical regression, ratio, exponential-ratio, Rao (1991), Smarandache et al. (2009), Shabbir and Gupta (2010), Grover and Kaur (2011), Grover and Kaur (2014) and Haq et al. (2017) estimators.

Aknolwedgement

Authors are grateful to the Editor and managing team for their support.

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