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Exploring Lomax-Gumbel {Fréchet} Distribution in the Bayesian Paradigm

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Abstract

This paper explores the Lomax-Gumbel {Fréchet} distribution in the Bayesian paradigm. The posterior distributions of the parameters are not attained in closed form, so the Lindley and the Tierney-Kadane approximation methods are used for the evaluation of Bayes estimators and associated posterior risks under uniform, Maxwell, and half logistic priors. A complete implementation of these two techniques is provided. Three loss functions are used. An extensive simulation study and two real-life data are provided to obtain and compare Bayes estimators in terms of prior distributions and loss functions. It is reported that the Bayes estimators obtained through the Tierney-Kadane method give better results than the Lindley approximation method, in terms of minimum posterior risks.

Key Words: *Lindley's approximation; Posterior distribution; Tierney-Kadane approximation methods; Lomax-Gumbel {Fréchet} Distribution.*

Introduction

A statistical distribution is proposed to describe real-life phenomena, provided probabilities of possible outcomes, and is used for the prediction of the same phenomenon. Many distributions and their generalizations are studied and derived, this process is continuous due to variation and diversity in the real world. The common feature of these generalized distributions is more parameters that improve a distribution's goodness of fit property. On the other side, estimation problems occur due to a large number of parameters.

The Gumbel, Fréchet, and Weibull are three basic models known as type-type-II, type-III, and I extreme value distributions. These are used for modeling of events related to earthquakes, astronomy, the maximum height of ocean waves, the strength of the financial market, etc.

The monumental work in developing generalized distributions is done by combining existing distributions. (Alzaatreh et al., 2013) introduced the *Transformed-Transformer* technique. In this method, the CDF of a continuous rv is transformed into CDF of generalized distribution through a functional form of CDF of another distribution. Modification of this method is proposed by (Alzaatreh et al., 2014) and derived generalized normal families of distributions using a new framework $T-R\ \{Y\}$. (Mahmoud et al., 2019) used this technique and proposed the Lomax-Gumbel {Fréchet} (L-G{F}) distribution, by considering T as Lomax, R as Gumbel, and Y as Fréchet random variate. The Parameters are estimated through the maximum likelihood method. The estimation in the Bayesian setup has not been addressed so far. In this research article Bayesian estimation for the parameters of L-G{F} distribution is presented.

The Bayesian analysis has drawn attention in the recent era. It updates prior knowledge about parameters with current information in the form of posterior distribution which plays a pivotal role.

Sometimes, in Bayesian analysis problem is faced in the evaluation of posterior estimates due to complex models or involving intractable integrals. Several numerical and approximation techniques are purposed for this purpose. (Jaheen, 2005) proposed the Bayesian estimation of a finite mixture of two exponential distributions using the Lindley approximation method. (Kundu, 2008) obtained a closed-form expression for the Bayes estimators and credible interval for the scale parameter of progressively censored Weibull distribution using Lindley approximation and Gibbs sampling procedure. (Abbas et al., 2013) obtained approximate Bayes estimators of Gumbel type-II distribution under different loss functions. (Nassar & Abo-Kasem, 2017) described the Bayesian estimation for both scale and the shape parameters of inverse Weibull distribution based on an adaptive type-II progressive hybrid censoring scheme using the Lindley approximation method. (Gearhart & Kasturiratna, 2018) analyzed complicated actuarial models with the Bayesian approach using Gibbs sampling. (Cockayne et al., 2019) used the Bayesian probabilistic numerical methods.

In this study, the posterior distributions of the parameters of the L-G{F} distribution are not in closed form. So, approximation techniques are utilized.

The main goal of the present article is to derive approximate Bayes estimators and associated posterior risk of L-G{F} distribution using two techniques that are Lindley approximation (Lindley, 1980) and Tierney-Kadane(T-K) (Tierney & Kadane, 1986). Estimators are obtained using informative and non-informative priors under square error loss function (SELF), weighted loss function (WLF), and precautionary loss function (PLF). Furthermore, the key motivation of the study is to suggest practitioners for analyzing complicated statistical models in the Bayesian paradigm. The rest of the paper is organized as follows;

We derived L-G{F} distribution with likelihood and log-likelihood functions. The maximum likelihood estimators and information matrix are defined here. The posterior distributions using informative and non-informative priors are derived. The Bayes point estimators and corresponding posterior risk functions are formulated. To check the performance of Bayes estimators, the Monte Carlo simulation study is performed and presented. Two real-life applications are used to compare Bayes estimates obtained from two approximation techniques based on risk function. Finally, the article is concluded.

The Lomax-Gumbel {Fr$\acute{e}$chet} Distribution

Let random variable X follows $L-G\ \{F\}$ model with parameters α and γ, the CDF and PDF of the distribution are defined as;

$$F(x) = 1-(1+\gamma e^{x})^{-\alpha}\ ,\ \ -\infty \le x \le \infty,\ \alpha,\gamma > 0 \tag{1}$$

$$f(x) = \alpha\gamma e^{x}(1+\gamma e^{x})^{-\alpha-1}\ ,-\infty \le x \le \infty,\ \alpha,\gamma > 0 \tag{2}$$

Using the inverse transformation technique, a random number of the distribution can be generated as;

$$X = ln\left[\frac{(1-U)^{-\frac{1}{\alpha}}-1}{\gamma}\right] \tag{3}$$

where U is a uniform random variate having a unit interval.

For a random sample $X_1, X_2, X_3, \ldots, X_n$ of size n from L-G{F} distribution, the likelihood and log-likelihood functions are;

$$L(\alpha,\gamma;x) = \alpha^{n}\,\gamma^{n}e^{\sum x}\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1} \tag{4}$$

$$ln\,L(\alpha,\gamma;x) = n\ln(a) + nln(\gamma) + \sum x - (\alpha-1)\sum_{i=1}^{n} ln(1+\gamma e^{x_i}) \tag{5}$$

Equating score functions to zero, two normal equations are obtained;

$$\frac{\partial lnL(\alpha,\gamma;x)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} ln(1+\gamma e^{x_i}) = 0$$

$$\frac{\partial lnL(\alpha,\gamma;x)}{\partial\gamma} = \frac{n}{\gamma} - (\alpha+1)\sum_{i=1}^{n} ln(1+\gamma e^{x_i})^{-1} = 0$$

These two equations are not in closed form expression, hence the MLEs of parameters α and γ can not be obtained directly. In this study, we used the Newton-Ralphson iterative process through the R-package *maxLik* for the estimation of MLEs.

The Fisher information matrix capture the information about parameters that a sample contains. The

variances of unbiased estimators can be obtained using the inverse of this matrix.
Let $\Delta = (\alpha,\gamma)'$, then 2×2 symmetry Fisher information matrix is defined as;

$$I(\Delta) = -E_{\Delta}\begin{bmatrix} U_{\alpha\alpha} & U_{\alpha\gamma} \\ U_{\gamma\alpha} & U_{\gamma\gamma} \end{bmatrix} \tag{6}$$

where

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2}\ , \qquad U_{\gamma\gamma} = -\frac{n}{\gamma^2} + (\alpha+1)\sum_{i=1}^{n} e^{2x_i}(1+\gamma e^{x_i})^{-2}$$

$$U_{\gamma\alpha} = -\sum_{i=1}^{n} e^{x_i}\,(1+\gamma e^{x_i})^{-1}$$

Bayesian Analysis

The Bayesian theory provides a better technique, which not only depends upon data but also some prior knowledge about model parameters. It gives better inferential solutions for small data and complex statistical models.

This section deals with different symmetry and asymmetry loss functions, informative and non-informative priors for the derivation of the Bayes estimators, and associated posterior risk of the parameters of L-G{F} distribution using the Lindley and T-K approximation techniques.

The Prior and posterior distribution

In Bayesian statistics, the prior information about parameters combines with the probability distribution of data and the resultant is posterior distribution. (Arnold & Press, 1983) mentioned that there is no way to describe that one prior is better than the other. In this study, we used a non-informative uniform prior and two informative priors (half-logistic and Maxwell) for the parameters α and γ of L-G{F} distribution.

(Laplace,1820) proposed that, when no information about parameters is available uniform distribution can be used as a prior. The joint uniform prior distribution for α and γ is;

$$\pi(\alpha,\gamma) \propto 1, \alpha,\gamma > 0 \tag{7}$$

Using the Bayes theorem, the joint posterior distribution of α and γ under uniform prior is;

$$\pi(\alpha,\gamma|\underline{x}) = \frac{L(\alpha,\gamma;x)\,\pi(\alpha,\gamma)}{\int_0^\infty \int_0^\infty L(\alpha,\gamma;x)\,\pi(\alpha,\gamma)\,d_\alpha d_\gamma}$$

$$= \frac{\alpha^n\gamma^n e^{\sum x}\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}}{\int_0^\infty \int_0^\infty \alpha^n\gamma^n e^{\sum x}\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}\,d_\alpha d_\gamma} \tag{8}$$

The use of informative prior leads to reducing the value of the risk function of the Bayes estimator because the incorporation of prior in the likelihood function(data) is equivalent to increasing the number of observations in the given sample. In this study, Maxwell distribution is used as prior of α and γ, that is;

$$\pi(\alpha) = \alpha^2 e^{-\frac{\alpha^2}{a_1}}, \quad \alpha, a_1 > 0$$

$$\pi(\gamma) = \gamma^2 e^{-\frac{\gamma^2}{a_2}}, \quad \gamma, a_2 > 0$$

The hyperparameters a_1 and a_2 are quantities that can be elicitated. (Aslam, 2003) elicits the values of hyperparameters through prior predictive distribution. Assuming the independence of prior, the joint prior distribution is;

$$\pi(\alpha,\gamma) \propto \alpha^2\gamma^2 exp\left(-\left(\frac{\alpha^2}{a_1}+\frac{\gamma^2}{a_2}\right)\right),\ a_1, a_2, \alpha, \gamma > 0 \tag{9}$$

By combining likelihood and joint prior distributions given in equations (4) and (9), the posterior distribution under Maxwell prior is;

$$\pi(\alpha,\gamma|\underline{x}) = \frac{\alpha^{n+2}\gamma^{n+2} e^{\left(\sum x - \frac{\alpha^2}{a_1} - \frac{\gamma^2}{a_2}\right)}\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}}{\int_0^\infty \int_0^\infty \alpha^{n+2}\gamma^{n+2} e^{\left(\sum x - \frac{\alpha^2}{a_1} - \frac{\gamma^2}{a_2}\right)}\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}\,d_\alpha d_\gamma} \tag{10}$$

Another informative prior used in this study is half logistic distribution;

$$\pi(\alpha) = \frac{e^{\alpha}}{(1+e^{\alpha})^{c+1}} \qquad \alpha > 0, c > 0$$

$$\pi(\gamma) = \frac{e^{\gamma}}{(1+e^{\gamma})^{d+1}} \qquad \gamma > 0, d > 0$$

where c and d are hyperparameters. The joint prior distribution of α and γ is;

$$\pi(\alpha, \gamma) = \frac{e^{\alpha+\gamma}}{(1+e^{\alpha})^{c+1}(1+e^{\gamma})^{d+1}} \quad c, d, \alpha, \gamma \geq 0 \qquad (11)$$

Posterior distribution under half logistic prior is;

$$\pi(\alpha, \gamma|\underline{x}) = \frac{\frac{\alpha^{n}\gamma^{n} exp(\sum x+\alpha+\gamma)\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}}{(1+e^{\alpha})^{c+1}(1+e^{\gamma})^{d+1}}}{\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{n}\gamma^{n} exp(\sum x+\alpha+\gamma)\prod_{i=1}^{n}(1+\gamma e^{x_i})^{-\alpha-1}}{(1+e^{\alpha})^{c+1}(1+e^{\gamma})^{d+1}}d_{\gamma}d_{\alpha}} \qquad (12)$$

Now inference of parameters α and γ based on posterior distributions given in equations (8), (10), and (12). Under different loss functions, Bayes estimators and associated posterior are derived in the next section.

Bayes Point Estimation

Decision theory suggests that loss function is the criteria to select the best estimator, risk is its average. Bayes estimators depend upon the choice of the loss function and are obtained by minimizing risk. For the Bayes point estimators of parameters α and γ three loss functions are considered in this study. That are SELF, WLF, and PLF.

Bayes estimators and corresponding posterior risks of any parameter θ under three loss functions defined in this study are presented in Table 1.

The Bayes estimators and corresponding posterior risk are obtained taking the posterior expectation of a function of the parameter, which is sometimes tedious to solve because of its complex integral form. In literature, various solutions are adopted to tackle this problem. (Geman & Geman,1984) introduced Gibbs sampling algorithm. (Kundu & Pradhan, 2009) used this technique for the construction of Bayes estimators and highest posterior density credible intervals for the parameters of generalized exponential distribution and also used Lindley approximation and importance sampling methods. (Abbas et al., 2012) suggested numerical methods for the derivation of the posterior distribution and Bayes estimators under informative and non-informative priors. (Nasir & Aslam, 2015) used the quadrature numerical integration technique for solving the posterior distribution.

The Bayes estimators and associated posterior risk of the parameters of L-G{F} distribution are derived by taking the expectation of a function of parameters under posterior distributions defined in equations(8), (10), and (12). But in all three cases, posterior expectation cannot be computed in explicit form due to the ratio of two integrals. Hence, Lindley and T-k approximation methods are used to approximate the expectation in the form of a single numerical result.

Table1: Different loss functions with their Bayes estimators and Posterior risk.

Loss functions	Bayes Estimators	Posterior Risk
SELF	$E(\theta\|\underline{x})$	$E(\theta^2\|\underline{x}) - \left(E(\theta\|\underline{x})\right)^2$
WLF	$\left(E(\theta^{-1}\|\underline{x})\right)^{-1}$	$E(\theta\|\underline{x}) - \left(E(\theta^{-1}\|\underline{x})\right)^{-1}$
PLF	$\sqrt{E(\theta^2\|\underline{x})}$	$2\left(\sqrt{E(\theta^2\|\underline{x})} - E(\theta\|\underline{x})\right)$

An extensive study is performed for the Bayes estimators and posterior risk using uniform, Maxwell, and half logistic priors under SELF, WLF, and PLF. Due to restriction in space, the theoretical results of Bayes estimators and corresponding posterior risks under half logistic prior are defined only.

The Bayes Estimators under Lindley Approximation

The posterior expectation for any function of parameters say $U(\alpha,\gamma)$ is defined as

$$E_{(\alpha,\gamma|\underline{x})}[U(\alpha,\gamma)] = \frac{\int_\alpha \int_\gamma U(\alpha,\gamma;x)\pi(\alpha,\gamma) d_\gamma d_\alpha}{\int_\alpha \int_\gamma L(\alpha,\gamma;x)\pi(\alpha,\gamma) d_\gamma d_\alpha} \tag{13}$$

$$= \frac{\int_\alpha \int_\gamma U(\alpha,\gamma;x)\pi(\alpha,\gamma) d_\gamma d_\alpha}{\int_\alpha \int_\gamma e^{\ln L(\alpha,\gamma;x)+\rho(\alpha,\gamma)} d_\gamma d_\alpha} \tag{14}$$

Where $\rho(\alpha,\gamma) = ln\pi(\alpha,\gamma)$

For a sufficiently large sample size, equation (14) can be evaluated using the Lindley approximation technique as;

$$E_{(\alpha,\gamma|\underline{x})}[U(\alpha,\gamma)] \approx U(\hat{\alpha},\hat{\gamma}) + \frac{1}{2}[u_{11}\sigma_{11} + u_{12}\sigma_{12} + u_{22}\sigma_{22}] + U_1\rho_1 + U_2\rho_2 + \frac{1}{2}[L_{30}\sigma_{11}U_1 + L_{21}(2\sigma_{12}U_1 + \sigma_{11}U_2) + L_{12}(\sigma_{22}U_1 + 2\sigma_{12}U_2) + L_{03}\sigma_{22}U_2] \tag{15}$$

$\hat{\alpha}$ and $\hat{\gamma}$ are MLEs of α and γ, σ_{ij} are *ij*-th elements of the inverse of Fisher information matrix and

$$u_1 = \frac{\partial U(\alpha,\gamma)}{\partial \alpha}, \quad u_2 = \frac{\partial U(\alpha,\gamma)}{\partial \gamma}, \quad u_{11} = \frac{\partial^2 U(\alpha,\gamma)}{\partial \alpha^2}, \quad u_{22} = \frac{\partial^2 U(\alpha,\gamma)}{\partial \gamma^2},$$

$$u_{12} = u_{21} = \frac{\partial^2 U(\alpha,\gamma)}{\partial \alpha\, \partial \gamma}$$

$$U_1 = u_1\sigma_{11} + u_2\sigma_{12} \quad , \quad U_2 = u_1\sigma_{21} + u_2\sigma_{22} ,$$

$$\rho = ln\pi(\alpha,\gamma) = (\alpha+\gamma) - (c+1)\, ln(1+e^{\alpha}) - (d+1) ln(1+e^{\gamma})$$

$$\rho_1 = \frac{\partial \rho}{\partial \alpha} = 1 - \frac{(c+1)e^{\alpha}}{(1+e^{\alpha})} \quad , \quad \rho_2 = \frac{\partial \rho}{\partial \gamma} = 1 - \frac{(d+1)e^{\gamma}}{(1+e^{\gamma})}$$

$$L_{30} = \frac{\partial^3 ln\, L(\alpha,\gamma;x)}{\partial a^3} = \frac{2n}{\alpha^3}, \quad L_{12} = \frac{\partial^3 ln\, L(\alpha,\gamma;x)}{\partial \alpha \partial \gamma^2} = \sum e^{2x}(1+\gamma e^{x_i})^{-2}$$

$$L_{21} = \frac{\partial^3 ln\, L(\alpha,\gamma;x)}{\partial \alpha^2 \partial \gamma} = 0, \quad L_{03} = \frac{\partial^3 ln\, L(\alpha,\gamma;x)}{\partial \gamma^3} = \frac{2n}{\gamma^3} - 2(\alpha+1)\sum e^{3x}(1+\gamma e^{x_i})^{-3}$$

The Bayes estimators of the parameter α using the Lindley approximation technique under specified loss functions are;

a) Square Error Loss Function (SELF)

$U(\alpha,\gamma) = \alpha$

$$\tilde{\alpha}_{LS} = \hat{\alpha} + \sigma_{11}\rho_1 + \sigma_{21}\rho_2 + \frac{1}{2}[L_{30}\sigma_{11}^2 + L_{12}(\sigma_{22}\sigma_{11} + 2\sigma_{12}^2) + L_{03}\sigma_{22}\sigma_{21}]$$

b) Weighted Loss Function (WLF)

$U(\alpha,\gamma) = \alpha^{-1}$

$$\tilde{\alpha}_{LW} = [[\hat{\alpha}^{-1} + \hat{\alpha}^{-3}\sigma_{11} - \hat{\alpha}^{-2}(\sigma_{11}\rho_1 + \sigma_{21}\rho_2)] - \frac{\hat{\alpha}^{-2}}{2}[L_{30}\sigma_{11}^2 + L_{12}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + L_{03}\sigma_{22}\sigma_{11}]]^{-1}$$

c) Precautionary Loss Function (PLF)

$U(\alpha,\gamma) = \sqrt{\alpha}$

$$\tilde{\alpha}_{LP} = \left[\hat{\alpha}^2 + \sigma_{11} + 2\hat{\alpha}(\sigma_{11}\rho_1 + \sigma_{12}\rho_2) + \hat{\alpha}[L_{30}\sigma_{11}^2 + L_{12}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + L_{03}\sigma_{22}\sigma_{11}]\right]^{\frac{1}{2}}$$

Similarly, Bayes estimators and posterior risk of parameters γ can be derived.

The Bayes Estimators under Tierney – Kadane's (T-K) Approximation

This subsection presents the approximate Bayes estimators of parameters of L-G {F} distribution through the T-K method. This technique provides an approximate solution to the ratio of two integrals. It solves numerators and denominators separately unlike the Lindley approximation. The posterior expectation given in equation (13) is evaluated as;

$$E_{(\alpha,\gamma|\underline{x})}[U(\alpha,\gamma)] = \frac{\int_\alpha\int_\gamma e^{n\left[\frac{1}{n}\ln U(\alpha,\gamma;x)+\frac{1}{n}\rho\right]}d_\gamma d_\alpha}{\int_\alpha\int_\gamma e^{n\left[\frac{1}{n}\ln L(\alpha,\gamma;x)+\frac{1}{n}\rho\right]}d_\gamma d_\alpha}$$
$$= \frac{\int_\alpha\int_\gamma e^{nl^*(\alpha,\gamma)}d_\gamma d_\alpha}{\int_\alpha\int_\gamma e^{nl(\alpha,\gamma)}d_\gamma d_\alpha}$$
$$\approx \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp[nl^*(\hat{\alpha}_{l^*},\hat{\gamma}_{l^*}) - nl(\hat{\alpha}_l,\hat{\gamma}_l)]$$

Where

$$l^*(\alpha,\gamma) = \frac{1}{n}lnU(\alpha,\gamma) + \frac{1}{n}lnL(\alpha,\gamma;x) + \frac{1}{n}\rho$$
$$l(\alpha,\gamma) = \frac{1}{n}lnL(\alpha,\gamma;x) + \frac{1}{n}\rho$$

$\sum^*$ and Σ are the negative inverse of the Hessian of $l^*(\alpha,\gamma)$ and $l(\alpha,\gamma)$ respectively. $\hat{\alpha}_{l^*}$and $\hat{\gamma}_{l^*}$ are the point of maxima of $l^*(\alpha,\gamma)$ similarly $\hat{\alpha}_l$ and $\hat{\gamma}_l$ for $l(\alpha,\gamma)$. This method involves the second derivative of functions. Contrarily, Lindley approximation utilizes the third derivative of the log likelihood function, which is tedious to solve in some cases.

The Bayes estimators of parameter α using half logistic prior under specified loss functions are;

a) Square error loss function (SELF)

$U(\alpha,\gamma) = \alpha$

$$\tilde{\alpha}(TKS) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[\left[ln\hat{\alpha}_{l^*} + lnL(\hat{\alpha}_{l^*},\hat{\gamma}_{l^*};x) + \rho_{l^*}\right] - \left[lnL(\hat{\alpha}_l,\hat{\gamma}_l;x)+\rho_l\right]\right]$$

b) Weighted loss function (WLF)

$U(\alpha,\gamma) = \alpha^{-1}$

$$\tilde{\alpha}(TKW) = \left[\sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[\left[1 - ln\hat{\alpha}_{l^*} + lnL(\hat{\alpha}_{l^*},\hat{\gamma}_{l^*};x) + \rho_{l^*}\right] - \left[lnL(\hat{\alpha}_l,\hat{\gamma}_l;x) + \rho_l\right]\right]\right]^{-1}$$

c) Precautionary loss function (PLF)

$U(\alpha,\gamma) = \sqrt{\alpha}$

$$\tilde{\alpha}(TKP) = \left[\sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[\left[2ln\hat{\alpha}_{l^*} + lnL(\hat{\alpha}_{l^*},\hat{\gamma}_{l^*};x) + \rho_{l^*}\right] - \left[lnL(\hat{\alpha}_l,\hat{\gamma}_l;x) + \rho_l\right]\right]\right]^{1/2}$$

Where

$$\rho_k = (\hat{\alpha}_k + \hat{\gamma}_k) - (c+1)ln\left(1 + e^{\hat{\alpha}_k}\right) - (d+1)ln\left(1 + e^{\hat{\gamma}_k}\right), k = l, l^*$$

Similarly, Bayes estimators and the posterior risk of parameters γ are derived.

Simulation Study

A Monte Carlo study is performed to obtain the Bayes estimators and associated posterior risk of α and γ using Lindley and T-K approximation methods. The purpose of this study is to check the performance of these estimators for sample sizes, priors, loss functions and to asses which approximation method performs better. Random sample of size n = 30, 50, 100, 120, 170, 250, 300, 400, 500 are generated from $L - G\{F\}$ distribution using generator given in equation (3).

It is assumed that $\alpha = 0.2$ and $\gamma = 0.1$. The MLEs of the parameters are obtained using Newton Ralphson iterative method and R-package *maxLik* is used. Bayes estimates and posterior risk using

uniform, half logistic, and Maxwell priors under SELF, WLF, and PLF are obtained using theoretical results derived above. All the process is repeated 1000 times and computation is performed using a programming routine in R-language.
The results are summarized in Tables 2 - 7. It is clear from these results that the performance of all Bayes estimators is well due to minimum values of posterior risk. Estimators obtained through the T-K method are approximately better than the Lindley method. Informative priors provide better estimates than non-informative prior. Its means, that when information about parameters is available, then it is better to use informative prior. It is observed for a large sample size that in the T-K method, values of risk in WLF and PLF are approximately equal to each other. Estimators under SELF are best due to minimum the risk.

Table 2: The Bayes estimates and posterior risks of α using a uniform prior

n	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T-K Method**		
30	0.20618	0.39091	0.21165	0.20576	0.19704	0.21112
	0.00253	0.18473	0.01094	0.00242	0.00871	0.01071
50	0.20553	0.34485	0.20865	0.20187	0.20092	0.20489
	0.00135	0.13932	0.00625	0.00128	0.00094	0.00604
100	0.20097	0.23342	0.20246	0.20208	0.19867	0.20336
	0.00061	0.03244	0.00296	0.00042	0.00340	0.00255
120	0.20078	0.22626	0.20201	0.20109	0.19961	0.20247
	0.00050	0.02548	0.00246	0.00053	0.00148	0.00275
170	0.20068	0.21686	0.20154	0.20035	0.19919	0.20178
	0.00035	0.01617	0.00172	0.00046	0.00116	0.00285
250	0.20026	0.21056	0.20084	0.20073	0.19954	0.20129
	0.00023	0.01029	0.00116	0.00022	0.00118	0.00113
300	0.20057	0.20902	0.20105	0.20190	0.19986	0.20170
	0.00019	0.00845	0.00097	0.00100	0.00204	0.00042
400	0.20007	0.20615	0.20043	0.19996	0.19925	0.20033
	0.00014	0.00608	0.00072	0.00014	0.00070	0.00074
500	0.20004	0.20489	0.20033	0.20077	0.20037	0.20116
	0.00011	0.00484	0.00057	0.00040	0.00299	0.00383

Table 3: The Bayes estimates and posterior risk of γ using uniform prior

n	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T-K Method**		
30	0.21975	0.16819	0.21298	0.33690	0.14080	0.46168
	0.00883	0.05156	0.01355	0.27855	0.19610	0.38926
50	0.15783	0.12974	0.15964	0.18624	0.12792	0.23519
	0.00052	0.02808	0.00565	0.03202	0.05831	0.09785
100	0.13154	0.11517	0.13580	0.13508	0.11662	0.14805
	0.00134	0.01637	0.00852	0.00451	0.01845	0.02594
120	0.12564	0.11203	0.12963	0.12843	0.11235	0.13859
	0.00120	0.01361	0.00798	0.00326	0.01607	0.02031
170	0.11782	0.10834	0.12112	0.11860	0.10815	0.12475
	0.00087	0.00948	0.00659	0.00169	0.01045	0.01229
250	0.11259	0.10613	0.11512	0.11246	0.10536	0.11625
	0.00062	0.00646	0.00506	0.00092	0.00710	0.00757
300	0.10954	0.10423	0.11172	0.10995	0.10405	0.11297
	0.00050	0.00531	0.00434	0.00061	0.00601	0.00604

400	0.10855	0.10453	0.11028	0.10862	0.10581	0.11084
	0.00039	0.00402	0.00346	0.00051	0.00283	0.00444
500	0.10609	0.10288	0.10751	0.10542	0.10246	0.10736
	0.00031	0.00320	0.00284	0.00040	0.00299	0.00383

Table 4: The Bayes estimates and posterior risk of α using Maxwell prior

SELF	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
Lindley Method			**T-K Method**		
0.18699	2.67617	0.19072	0.33845	0.26273	0.28324
0.00157	2.48918	0.00745	4.35102	0.07571	0.11042
0.19249	0.32386	0.19511	0.30845	0.23388	0.24664
0.00106	0.13137	0.00524	4.75259	0.07456	0.12362
0.19598	0.22661	0.19737	0.21952	0.24309	0.22078
0.00055	0.03062	0.00276	0.00058	0.02357	0.00251
0.19662	0.22081	0.19778	0.21499	0.13893	0.21599
0.00046	0.02419	0.00232	0.00050	0.92393	0.00199
0.19837	0.21420	0.19920	0.21047	0.20873	0.21136
0.00033	0.01583	0.00166	0.00037	0.00173	0.00177
0.19852	0.20873	0.19909	0.20705	0.20588	0.20765
0.00022	0.01020	0.00113	0.00024	0.00116	0.00119
0.19884	0.20714	0.19932	0.20624	0.20526	0.20672
0.00019	0.00830	0.00094	0.00020	0.00097	0.00096
0.19889	0.20497	0.19925	0.20442	0.20366	0.20478
0.00014	0.00607	0.00071	0.00014	0.00075	0.00072
0.19892	0.20369	0.19920	0.20431	0.20373	0.20460
0.00011	0.00477	0.00057	0.00012	0.00057	0.00059

Table 5: The Bayes estimates and posterior risk of γ using Maxwell prior

n	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T-K Method**		
30	0.32613	0.02279	0.26335	0.08251	0.11035	0.09443
	0.07581	0.30334	0.12556	0.08094	0.02783	0.02383
50	0.22800	0.86331	0.20574	0.08691	0.06289	0.09803
	0.01571	0.63530	0.04452	0.00459	0.02401	0.02224
100	0.16198	0.16665	0.15814	0.09025	0.07824	0.09809
	0.00162	0.00467	0.00768	0.00169	0.01200	0.01567
120	0.14858	0.14755	0.14678	0.09006	0.08079	0.09662
	0.00069	0.00102	0.00359	0.00137	0.00926	0.01311
170	0.13388	0.12966	0.13406	0.09470	0.08651	0.09923
	0.00003	0.00421	0.00036	0.00096	0.00818	0.00905
250	0.12203	0.11784	0.12298	0.09586	0.09008	0.09895
	0.00024	0.00419	0.00190	0.00064	0.00577	0.00618
300	0.11853	0.11467	0.11959	0.09722	0.09228	0.09984
	0.00026	0.00385	0.00213	0.00054	0.00494	0.00523
400	0.11506	0.11182	0.11613	0.09845	0.09464	0.10042

	0.00025	0.00323	0.00215	0.00041	0.00380	0.00394
500	0.11249	0.10976	0.11349	0.09694	0.09395	0.09849
	0.00023	0.00273	0.00199	0.00031	0.00299	0.00310

Table 6: The Bayes estimates and posterior risk of α using half logistic prior

n	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T-K Method**		
30	0.20170	0.19476	0.20610	0.14695	0.14207	0.14936
	0.00181	0.00693	0.00881	0.00072	0.00487	0.00483
50	0.20075	0.19561	0.20369	0.14463	0.14171	0.14608
	0.00122	0.00514	0.00586	0.00042	0.00291	0.00291
100	0.20026	0.19750	0.20173	0.14305	0.14162	0.14375
	0.00060	0.00275	0.00293	0.00020	0.00143	0.00140
120	0.20050	0.19817	0.20172	0.14305	0.14186	0.14366
	0.00050	0.00232	0.00245	0.00017	0.00119	0.00121
170	0.20045	0.19879	0.20131	0.14201	0.14116	0.14243
	0.00034	0.00165	0.00172	0.00012	0.00084	0.00084
250	0.20096	0.19981	0.20154	0.14177	0.14120	0.14206
	0.00023	0.00114	0.00117	8.1e-05	0.00057	0.00057
300	0.19962	0.19867	0.20010	0.14170	0.14120	0.14191
	0.00019	0.00094	0.00096	6.1e-05	0.00049	0.00043
400	0.20023	0.19952	0.20059	0.14149	0.14109	0.14167
	0.00014	0.00071	0.00072	4.9e-05	0.00039	0.00035
500	0.20031	0.19974	0.20060	0.14141	0.14110	0.14156
	0.00011	0.00057	0.00057	4.1e-05	0.00031	0.00029

Table 7: The Bayes estimates and posterior risk of γ using half logistic prior

n	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T-K Method**		
30	0.10752	0.10093	0.15691	0.05538	0.04797	0.05917
	0.34231	0.00659	0.02281	0.00043	0.00740	0.00759
50	0.13833	0.10707	0.14923	0.05149	0.04709	0.05373
	0.00316	0.03125	0.02044	0.00023	0.00439	0.00448
100	0.12251	0.10575	0.12863	0.04895	0.04726	0.05057
	0.00190	0.01675	0.01224	0.00013	0.00169	0.00323
120	0.11866	0.10473	0.12388	0.04802	0.04710	0.04986
	0.00149	0.01392	0.01043	0.00013	0.00091	0.00368
170	0.11286	0.10322	0.11674	0.04803	0.04675	0.04869
	0.00100	0.00964	0.00775	6.4e-05	0.00128	0.00132
250	0.10887	0.10239	0.11165	0.04761	0.04673	0.04805
	0.00066	0.00648	0.00556	4.1e-05	0.00088	0.00087
300	0.10884	0.10337	0.11124	0.04725	0.04652	0.04762
	0.00056	0.00546	0.00480	3.4e-05	0.00073	0.00072
400	0.10639	0.10234	0.10822	0.04720	0.04664	0.04747
	0.00041	0.00405	0.00366	2.5e-05	0.00056	0.00053
500	0.10397	0.10078	0.10544	0.04709	0.04664	0.04730
	0.00031	0.00318	0.00293	2e-05	0.00045	0.0004

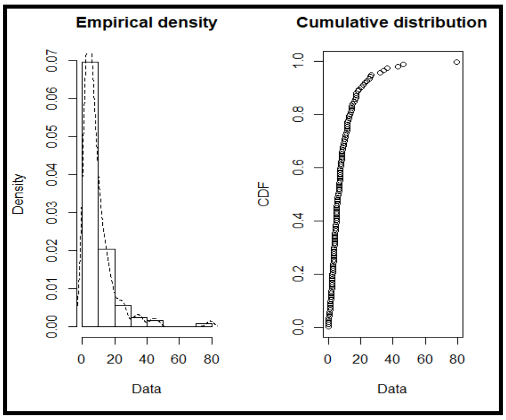
Real-Life Applications

To illustrate the performance of Bayes estimators, two real-life data sets are taken. From these data sets the Bayes estimates under SELF, WLF and PLF are evaluated using Lindley and T-k approximation techniques. Uniform, Maxwell, and half logistic priors for α and γ are used. All the parameters are estimated using R-language.

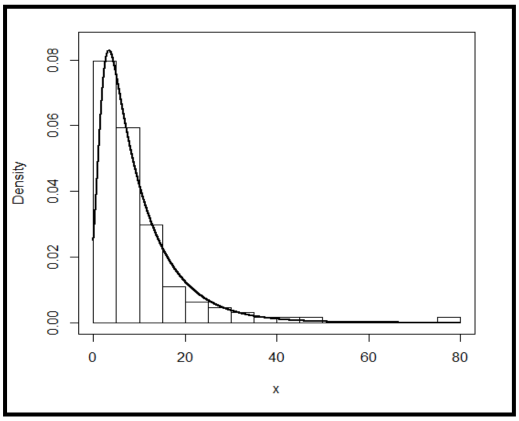
Data Set 1:

Bladder cancer is a disease in which the growth of abnormal tissue (tumor), develops in the bladder lining. Tobacco and smoking are its main causes. painless blood in the urine or painful urination is the main symptom. This cancer is treatable with surgery, chemotherapy, and radiation. The data set is about the remission time of 128 bladder cancer patients and it is previously used by (Lee & Wang, 2003; Zea et al.,2012; Lemonte & Cordeiro, 2013; Tahir et al. 2018).

Figure 1 (a) presents the empirical density and cumulative distribution of the data, and (b) represents the goodness of fit curve of L-G{F} distribution to the bladder cancer data. Table 8 shows the Bayes estimates and values of posterior risk for bladder cancer data.



(b)



(a)

Figure 1:

(a)

Empirical and cumulative distribution function of data set 1 (b) goodness of fit curve of L-G{F} distribution with the data

Data set 2:

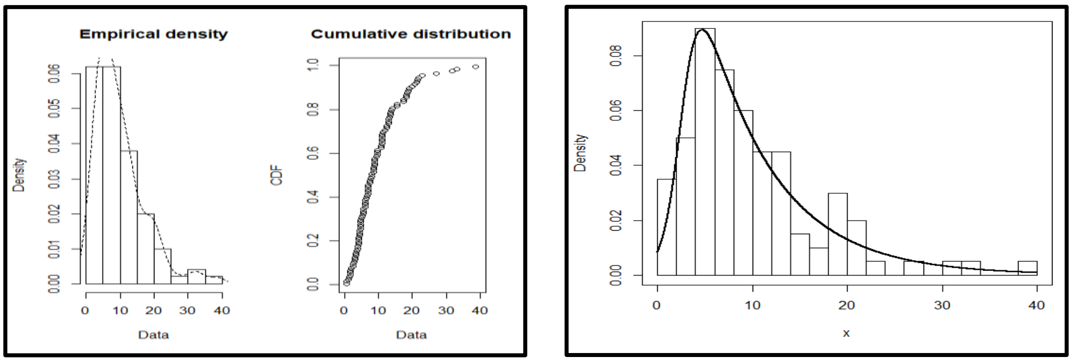
This data set shows the weighting time (in minutes) before service of 100 bank customers and it is previously used by (Ghitany et al., 2008; Shanker et al., 2015). Figure 2 (a) shows the empirical density and cumulative distribution of data set 2 and (b) is histogram/ pdf for L-G{F}distribution of data set to the distribution. Table 9 shows the Bayes estimates and posterior risks for data set 2.

Results of real-life data set 1 and 2 are reported in Tables 8 and 9. It is clear from the results that all the Bayes estimators perform well due to the minimum values of the risk function. The performance of estimators using the T-K method is better than Lindley's. The values of risk is minimum under informative priors than non-informative prior. Its means if information about parameters is available, then it is better to use informative prior rather than non-informative prior. By comparing loss functions, SELF provides minimum values of risk function as compared to WLF and PLF.

Table 8: The Bayes estimates and posterior risks using informative and non-informative priors under SELF, WLF, and PLF of data set 1

Parameter	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T – K Method**		
			Uniform Prior			
α	0.12103	0.11991	0.12161	0.12114	0.11995	0.12156
	0.00013	0.00112	0.00115	0.00010	0.00118	0.00084
γ	0.32830	0.29247	0.33821	0.33693	0.29103	0.36523
	0.00660	0.03582	0.01982	0.04590	0.05659	0.01987
			Maxwell Prior			
α	0.11963	0.11857	0.12020	0.12665	0.12617	0.12763

	0.00013	0.00105	0.00112	0.00024	0.00047	0.00195
γ	0.39546	0.40163	0.38794	0.24201	0.21309	0.26039
	0.00588	0.00617	0.001503	0.02892	0.03675	0.00923
			Half logistic Prior			
α	0.12337	0.12222	0.12394	0.11143	0.11129	0.11218
	0.00013	0.00115	0.00112	0.00016	0.00014	0.00150
γ	0.26583	0.23346	0.28425	0.06317	0.06109	0.06446
	0.01013	0.03237	0.03684	0.00016	0.00207	0.00259



(a)

(b)

Figure 2: (a) Empirical and cumulative distribution function of data set 2 (b) goodness of fit curve of L-G{F} distribution with the data set 2

Table 9: The Bayes estimates and posterior risk using informative and non-informative priors under SELF, WLF, and PLF of data set 2

Parameter	**SELF**	**WLF**	**PLF**	**SELF**	**WLF**	**PLF**
	Lindley Method			**T – K Method**		
			Uniform Prior			
α	0.13403	0.13237	0.13490	0.13406	0.13159	0.13485
	0.00023	0.00166	0.00174	0.00021	0.00247	0.00156
γ	0.08796	0.07530	0.09064	0.09210	0.07438	0.010380
	0.00047	0.01266	0.00534	0.01772	0.02388	0.00229
			Maxwell Prior			
α	0.13098	0.12954	0.13180	0.14373	0.14273	0.14490
	0.00021	0.00144	0.00163	0.00033	0.00099	0.00234
γ	0.11180	0.12504	0.10686	0.05784	0.04798	0.06396
	0.00108	0.01324	0.00989	0.00985	0.01223	0.00074
			Half logistic Prior			
α	0.13401	0.13235	0.13489	0.11424	0.11258	0.11486
	0.00023	0.00166	0.00174	0.00014	0.00165	0.00123
γ	0.08687	0.07395	0.089820	0.03594	0.03336	0.03720
	0.00052	0.01292	0.00591	9.23e-05	0.00258	0.00252

Conclusion

In this paper, the Bayesian estimation of Lomax-Gumbel {Fréchet} distribution using three symmetry and asymmetry loss functions are presented. We used informative and non-informative priors for the unknown parameters of the distribution. The posteriors distributions are not in closed form, so the explicit form of the Bayes estimators is not possible. We adopted two approximation techniques, Lindley and T-K for the derivation of approximate Bayes estimators and associated posterior risks. Theoretical results are illustrated through a simulation study, it is observed that

Bayes estimators obtained from two techniques perform well. When the sample size increases values of risk function decrease. The estimators under informative priors perform better than non-informative due to the high rate of decreasing posterior risk. Two real-life data sets are analyzed. It is observed that in both cases, estimators under informative priors provide minimum values of the risk function. The estimators obtained through the T-K method are better than the Lindley approximation method. Minimum values of posterior risk are obtained through estimators under SELF.

It is hoped that practitioners can estimate the parameters of the complex model through proposed Bayesian techniques.

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