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Ritz Vectors-Based Deflation Preconditioner for Linear System with Multiple Right-Hand Sides

Muhammad Humayoun[1*], S.M Aqil Burney[2], A.H. Sheikh[2], and Abdul Ghafoor[3]

[1]Department of Computer Science, Institute of Business Management, Karachi, Pakistan
[2]Mathematics and Statistics, Institute of Business Management Karachi, Pakistan
[3]BS & RS, Faculty of Science, QUEST, Nawabshah, Pakistan
*Corresponding Author: *humayoun.qureshi@iobm.edu.pk*

Abstract

Computational mathematics have many tools to solve the large systems of equations which may be linear or nonlinear. Iterative methods are used to solve the nonsymmetric definite system of linear equations like Krylov methods. Linear systems with multiple right-hand sides find application in many areas of engineering and science. Considering generality and indefiniteness, Krylov subspace methods are frequently used for such problems. However, problem with system with multiple right-hand side vectors requires constructing subspace for every right-hand side. GMRES produces Ritz vectors, approximation to eigenvectors during iterations. These Ritz values, recycled vectors are used in Krylov solve while solving system with second and subsequent right-hand side vectors. This is applied as a deflation preconditioner to GMRES. The numerical results show that the computational time, residuals and number of iterations is reduced as compare to simple GMRES. Deflation technique with Ritz vectors is not expensive as compare to the GMRES and as well as exact eigenvectors deflations.

Keywords: GMRES Method, Ritz Vector deflation, Eigenvalue deflation, Reduces CPU time, Residual.

Introduction

The system of linear equations was introduced in Europe in 1637 by Renee Descartes in Geometry. This new geometrical system, now called Cartesian geometry. In the linear system the equations are describe in the form of lines and plans, and if the lines and planes are intersecting each of them the solution of system of equations are exist, the more times the lines and planes intersect, the more solutions of linear system forms, which means the number of solutions is the numbers of the intersection of lines and planes in a system of equations. It is difficult to define the importance of linear algebra. Linear algebra is used everywhere like calculus, statistics etc. But not in a visible way for example, in statistics the Statisticians use both systems (linear and nonlinear system of equations) to predict what the future holds. If you have a lot of data about selective events you can predict what will happen in the future about the event and you can plan it, also you can determine the line or the best fit curve. In mathematics and physics, a very well-known equation named Poisson equations is used to find the electric field force and Poisson equations area type of elliptic partial differential equation of general mathematics in conceptual physics. As we know the potential field is the solution of Poisson equations and it is caused by the distribution of mass density with known potential field and calculating gravitational field force. It is obtaining by after the generalization of Laplace equation. There are three possibilities of the solution of a linear system: these are specified as they have one solution, they have multiple solutions (infinite solutions) they have no solution. To find these solutions we have methods to find it. For example, some basic methods such as graphing, substitution method and elimination method. There are many studies and application on the linear system such as some well-known studies and application are multiple right-hand side method now the recent work on this study is used in solving a large linear system of equations and the basic work of this system is in the Antenna designing also the system use in the GMRES method and so on. We can solve the linear equations system having multiple right-hand sides by many different methods, some of them are given as follows. We can reduce the prescribed linear system with multiple sides together to row echelon form or reduced row echelon form. Another method could be by inverting the coefficient matrix and finding multiple solutions by multiplying the inverse of the coefficient matrix with all RHS solutions one by one. We can solve the LS with multiple RHS solutions together by using Cramer's Rule, and it could be possible to easily solve for multiple solutions if we assume the RHS generally like (b1, b2, b3) and finding all determinants need to be solved in a general way, then we have to just plug in the different values of and could solve it by Cramer's rule. Linear systems that having multiple right-hand side vectors have attracted researchers recently, considering their growing applications. The author develops a new method with the help of global least square method, the proposed method has certain advantages, In this method, there is no need to find the dimension of a subspace and approximated solutions for the system, and here we can compute the residuals together very cheaply at each phase of the algorithm since the author upgrade them with short-term recurrence, in this paper results, shows that, the convergence criteria are very clear and the proposed method is more convergent than others (Karimi 2012). The linear system with left-right preconditioners are studied in this paper. The algorithm RL-PGLS is used to solve the equation matrix. Here the author uses this algorithm to the approximate generalized inverse of a close singular square matrix. The algorithm is the R-PGLS algorithm. After doing the numerical computations we can see the RL-PGLS and R-PGLS algorithms are extra effective as compare to the GL-LSQR algorithm (Karimi 2016). In this study, there is a work on deflated GMRES method is used to solve a large linear system of equations, also it is concluded that this method is faster than its other extensions (Bissuel et al. 2016). This article is describing the methods that can be added to the Algorithmic cryptanalysis toolbox, methods described in this the article should be entered in the toolbox Algorithmic cryptanalysis, which will help other algorithms to solve nonlinear equations System (Raddum and Semaev 2008). A new family of global A-biorthogonal methods are discussed here and is done by use of small two-term recurrences and formal orthogonal polynomials (Zhang, Dai, and Zhao 2011). Here author develops a new method with the generalization of the Newton-like method after the modification of the right-hand side vector for NCPs. In the development of the method, the author use semi-smooth reformulation with classical Minty function and the generalized Jacobian (Krejić, Lužanin, and Rapajić 2006). Here we have a new way to solve the system of nonsingular equations. A method is a form of a newton-like method. Initially there approximates Jacobian matrix, and also if we do LU factorization and modify it for right-hand side vector at each iteration, then we can construct it more-cheaper computationally as compared to the Newton method. The theme is to use the appropriate relaxation parameters and modify only those components which are only right-hand vectors (Krejić and Lužanin 2001). For three-block Krylov iterative methods, the author works to compare all methods to make new variants so that a large linear system of equations could be solved with multiple right-hand sides, and for this purpose author uses block Lanczos processes to explain three-block Krylov iterative methods. Numerically he proved the proposed the algorithm is more effective and cheaper than other algorithms for linear system of equations (Esghir et al. 2017). The authors analyze the set of Krylov projection methods since concentration is on a specific conjugate gradient (CG) implementation by Smith, Peterson, and Mittra to solve a linear system of equation with multiple right-hand sides (Chan and Wan 1997). The author makes an iterative method is used to solve the system of a linear equation and the method is called GMRES, the new algorithm has many advantages over the generalized conjugate residual method (GCR) and also ORTHODIR (Saad and Schultz 1986). Youcef Saad gives an iterative method to solve a large sparse nonsymmetric linear system of equations that enhance Manteuffel's adaptive Chebyshev method with a conjugate gradient-like method (Elman, Saad, and Saylor 1986). An application problem on a linear system with multiple right-hand side vector is ECGI (Electrocardiographic Imaging). The author uses an iterative method whose name is the GMRES method which replaces the Tikhonov solution. We because Tikhonov solution has missing localized potential features in some cases but GMRES recovered them (Ramanathan et al. 2003). A superfast iterative method to solve a linear system with multiple right hand side vector by use of the original method (Mikrin et al. 2018). A new method with deflation is used to solve a nonsymmetric system of linear equations with multiple right-hand sides. GMREs method with deflation is used to solve the nonsymmetric linear system with multiple right-hand sides. Here deflation is based on eigenvectors this is an alternate of the block GMRES method. Here they use selective eigenvectors to find the minimal residuals (Morgan and Wilcox 2004). The iterative methods are used for solving large scale linear system of equations most of the solvers are preconditioned Krylov (sub)space, solvers (Gutknecht 2007). Krylov space is an iterative method and it is used to solve a non-symmetric linear system of equations is based on the degree to polynomial equations which are lies in coefficient it starts from A^0=I and I is an identity matrix (Ford 2015).

As explained, quite a few solvers for the linear systems have been proposed. Some are truly classic and some are advanced, physics oriented and depends upon characteristics of coefficient matrices. Linear system with multiple right-hand sides, where same coefficients matrix is used, have recently been used in many applications and has been a challenging problem in terms of memory and computation complexity. The linear system (LS) with multiple right-hand side vectors can be read as AX=B

$$B = [b_1\ b_2\ b_3 \dots b_k]$$

Where;

$$Ax_1 = b_1, \qquad Ax_2 = b_2, \qquad Ax_3 = b_3, \dots, Ax_k = b_k.$$

Having same coefficient matrix to number of linear systems, to use and recycle information from first solve by any iterative method has been a natural attention by many researchers

Deflation as Preconditioner

The deflation is a useful technique for a nonsymmetric large system of linear equations, sparse or dense matrix. Deflation provides us with different types of advantages for slow conditions systems which give results very slowly and the iterations are very large and they required large memory and we want to a variant to restart the iterations by ignoring all and use as block Krylov subspace and convert a new deflated method with the least residuals. A simple methodology to keep important information to deflate limited eigenvalues when their eigenvectors are converged well approximated by the end of the cycle (Sheikh, Vuik, and Lahaye 2009). The iterative method to find the solution of the linear system, in the coefficient matrix the eigenvalues are affected poorly for ominous eigenvalues deflation (Sheikh 2014, 20014) for symmetric positive definite system (SPD) is used to improve conditions with Conjugate gradient method deflation of an Eigen space withdraws for the corresponding eigenvalues without affecting iterations (Sheikh et al. 2016).

There are many methods to solve the system of the linear equation the but as Arnoldi's method and Conjugate gradient method are the solvers of non-symmetric positive the definite linear system both methods are not able to solve non-symmetric the negative definite linear the system, the GMRES method is able to solve negative definite nonsymmetric linear system of equation In GMRES number of iterations are very large and the time duration for each iteration is huge for reducing both these we create a deflation "D". So, apply GMRES method for first iteration, and stop it. And store all ritz vector and use to find deflation "D". Multiply D with linear system and after this apply GMRES method on linear system of deflation D.

Algorithms for GMRES and GMRES with deflation are given below.

$$x_i = GMRES(A, b_i)$$

A and $b_i$are inputs for GMRES method and $x_i$is our output.

During solving GMRES iterations, produce Ritz Vectors (approximate eigen vectors).

Consider a system $Ax = b$ and suppose and $= \{b_0,\ b_1,\ b_2,\ b_3, \dots,\ b_n\}$ $n \in \mathbb{W},\ \ b = f$

so, the linear system is equal to $Ax = f$

- Start: Choose initial guess x_0 and compute $r_0 = f - Ax_0$ and $v_1 = \frac{r_0}{\|r_0\|}$
- Iterate: for $j = 1,2,3,4, \dots, k, \dots$, until satisfied do:
- Choose $h_{i,j}$
- $h_{i,j} = (Av_j, v_i), i = 1,2,3,4, \dots, j,$

  After finding $h_{i,j}$ compute $\hat{v}_{j+1}$ by the use of $h_{i,j}$
- $\hat{v}_{j+1} = Av_j - \sum_{i=1}^{j} h_{i,j} v_i,$

  Find norm of $\hat{v}_{j+1}$which is equal to $h_{j+1,j}$ as
- $h_{j+1,j} = \| \hat{v}_{j+1} \|$, and get Ritz vectors $v_{j+1} = \frac{\hat{v}_{j+1}}{\|\hat{v}_{j+1}\|}$

It can be written in this form $v_{j+1} = \frac{\hat{v}_{j+1}}{h_{j+1,j}}$. We use Ritz vectors for construct deflation and it is denoted as

$$D = I - AQ,$$

where D is deflation, A is coefficient matrix and I is identity matrix and

$$Q = ZE^{-1}Z^T$$

and

$$E = Z^T AZ$$

Suppose $v_1\ v_2\ v_3 \dots$ are Ritz vectors and Z is, matrix, consisting of Ritz vectors as columns $Z = [v_1\ v_2\ v_3 \dots]$

Now this deflation can be used to solve $Ax_2 = b_2$, and so on.

Constructing Deflation pre-conditioner

The faster convergence has been achieved by the preconditioned technique of Krylov subspace methods through some important information. A good preconditioned technique has constructed, as it works on every iteration. So, for this purpose, we have required a flexible Krylov method. In many problems preconditioners does not give desired results (Shaikh et al. 2019; Siyal et al. 2019; Sheikh 2014). In many cases, the spectrum stagnate converges sufficiently. In order to explain deflation based GMRES solver, the simplified GMRES algorithm is presented;

- Start: Choose initial guess x_0 and compute $r_0 = f - Ax_0$ and $v_1 = \frac{r_0}{\|r_0\|}$
- Iterate: for $j = 1,2,3,4, \dots, k, \dots$, until satisfied do:
- $h_{i,j} = (Av_j, v_i), i = 1,2,3,4, \dots, j,$
- $\hat{v}_{j+1} = Av_j - \sum_{i=1}^{j} h_{i,j} v_i,$
- $h_{j+1,j} = \| \hat{v}_{j+1} \|,\ \ and$
- $v_{j+1} = \frac{\hat{v}_{j+1}}{h_{j+1,j}},$

- From the approximation solution:
- $x_k = x_0 + v_k y_k$; where y_k minimizes $J(y) = \| \beta e_1 - \bar{H}_k y \|$, $y \in R^k$

Here we are describing Arnoldi's method to compute the eigenvalues of nonsymmetric matrices(Elman, Saad, and Saylor 1986)

Choose an arbitrary vector $v_1$there norm is equal to 1 $\| v_1 \|_2 = 1$and use the Krylov subspace

$k_m = span\{v_1, Av_1, \dots, A^{m-1}v_1\}$ use for approximating the eigen values of A it gives a set of eigen values estimator $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$

So there exist $u_i \in K_m$ $i = 1, \dots, m$ for which

$$(Au_i - \lambda_i u_i, v) = 0, \quad i = 1, \dots, m$$

$\forall\, v \in k_m$ and it construct an orthogonal matrix $V_m = \{v_1, \dots, v_m\}$ whose columns are $\{v_j\}_{j=i}^{m}$ $span\ K_m$.

GMRES Algorithm with Eigenvector Deflation

Here we are describing Arnoldi's method to compute the eigenvalues of non-symmetric matrices (Elman, Saad, and Saylor 1986)

Choose an arbitrary vector $v_1$there norm is equal to 1 $\| v_1 \|_2 = 1$and use the Krylov subspace

$k_m = span\{v_1, Av_1, \dots, A^{m-1}v_1\}$ use for approximating the eigen values of A it gives a set of eigen values estimator $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$.

1) Start: Choose initial guess x_0 and compute $r_0 = f - Ax_0$, eigenvalues for A and $v_1 = \frac{r_0}{\|r_0\|}$
2) Iterate: for $j = 1,2,3,4, \dots, k$,…, until satisfied do:
   - $h_{i,j} = (Av_j, v_i), i = 1,2,3,4, \dots, j,$
   - $\hat{v}_{j+1} = Av_j - \sum_{i=1}^{j} h_{i,j} v_i,$
   - $h_{j+1,j} = \| \hat{v}_{j+1} \|$, and
   - $v_{j+1} = \frac{\hat{v}_{j+1}}{h_{j+1,j}},$
3) Evaluate the Deflation "D" with eigenvalues
   - Chose some v from Z eigenvalue matrix
   - $Z = [v_1\ v_2\ v_3 \dots]$
   - Find Z^T to evaluate E, $E = Z^T AZ$
   - Use E to find Q, $Q = ZE^{-1}Z^T$
   - Use All these to ake D Deflation, $D = I - AQ$
   - Use Deflation D with Linear system, $DAX = Db$
4) From the approximation solution:
   - $x_k = x_0 + v_k y_k$; where y_k minimizes $J(y) = \| \beta e_1 - \bar{H}_k y \|$, $y \in R^k$

So there exist $u_i \in K_m$ $i = 1, \dots, m$ for which $(Au_i - \lambda_i u_i, v) = 0$, $i = 1, \dots, m$, $\forall\, v \in k_m$ and it constructs an orthogonal matrix $V_m = \{v_1, \dots, v_m\}$ whose columns are

$\{v_j\}_{j=i}^{m}$ $span\ K_m$.

GMRES Algorithm with Ritz vectors deflation (Elman, Saad, and Saylor 1986)

1) Start: Choose initial guess x_0 and compute $r_0 = f - Ax_0$ and $v_1 = \frac{r_0}{\|r_0\|}$
2) Iterate: for $j = 1,2,3,4, \dots, k, \dots$, until satisfied do:
    - $h_{i,j} = (Av_j, v_i), i = 1,2,3,4, \dots, j,$
    - $\hat{v}_{j+1} = Av_j - \sum_{i=1}^{j} h_{i,j} v_i,$
    - $h_{j+1,j} = \| \hat{v}_{j+1} \|$, and
    - $v_{j+1} = \frac{\hat{v}_{j+1}}{h_{j+1,j}},$
    - Stop after iteration 1 and store v_{i+1}
3) Evaluate the Deflation "D"
    - Chose some v from v_{j+1}
    - $Z = [v_1\ v_2\ v_3 \dots]$
    - Find Z^T to evaluate E , $E = Z^T AZ$
    - Use E to find $Q, Q = ZE^{-1}Z^T$
    - Use All these to make D Deflation, $D = I - AQ$
    - Use Deflation D with Linear system $DAX = Db$
4) From the approximation solution:

$x_k = x_0 + v_k y_k$; where y_k minimizes $J(y) = \| \beta e_1 - \bar{H}_k y \|$, $y \in R^k$
Here we are describing Arnoldi's method to compute the eigenvalues of non-symmetric matrices. Choose an arbitrary vector $v_1$there norm is equal to 1 $\| v_1 \|_2 = 1$and use the Krylov subspace

$k_m = span\{v_1, Av_1, \dots, A^{m-1}v_1\}$ use for approximating the eigen values of A it gives a set of eigen values estimator $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$

So there exist $u_i \in K_m$ $i = 1, \dots, m$ for which $(Au_i - \lambda_i u_i, v) = 0,$ $i = 1, \dots, m$ and $\forall\ v \in k_m$ and it construct an orthogonal matrix $V_m = \{v_1, \dots, v_m\}$ whose columns are $\{v_j\}_{j=i}^{m}$ $span\ K_m$(Elman, Saad, and Saylor 1986)

<u>Algorithm to extract Ritz vectors from GMRES:</u>

For $j = 1,2, \dots do$, solve r from $P_r = b - Ax^0$, then

$v^1 = \frac{r}{\|r\|_2}$ and

$$s = \| r \|_2\ e_1$$

For $i =\ 1,2, \dots, m\ do$, Solve w from $P_w = Av^i$,

for $k = 1,2, \dots, i\ do$

$$h_{k,i} = (wh, v^k)$$

$$w = w - h_{k,i} v^k$$

end

end

For Compute vector v which minimize $\| w - H_m y \|$ and $x^m = x^0 + y_1 \times v^1 + \cdots + y_m \times v^m$. If this satisfies then stop, else set $x^0 = x^m$.

Results with eigenvectors and Ritz vectors

The two dimensional Helmholtz equation in Cartesian coordinates with boundary $\partial\nabla$ for $u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} - k^2 u(x, y) = f(x, y).$$

To check the proposed algorithm for problem $f_1(x, y) = -6xy^2(x^2 + 2y^2) + 5x^3y^2$ experiment is performed by using of GMRES with one right hand side function

$$K = \frac{2\pi f}{\lambda}$$

Table 1: GMRES Iterations and computational time for different grid size n and wave number K.

K	N	GMRES Iterations	CPU Time in second
10	16	57	1.3753
20	32	145	1.4275
30	48	240	8.7847
40	64	343	34.6109
50	80	436	93.7220

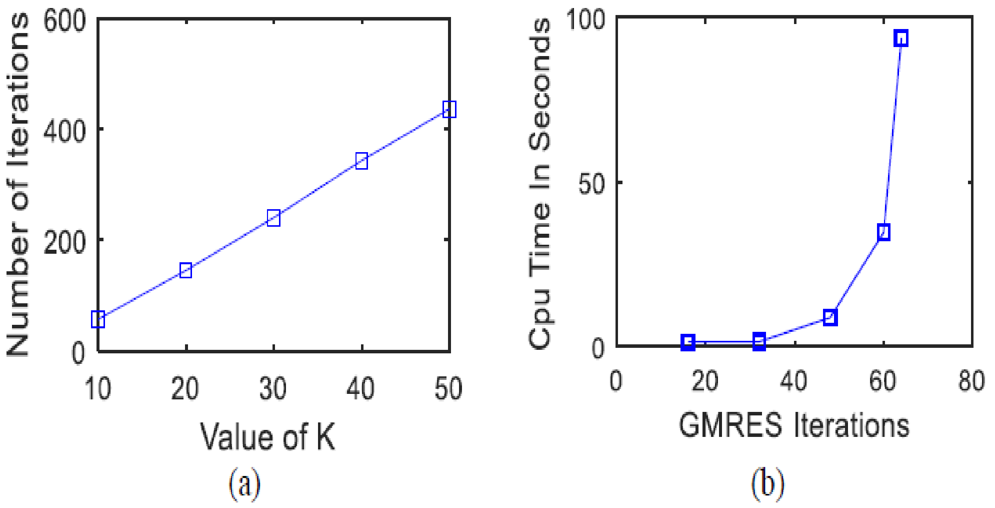

Figure 1: (a) of GMRES iterations with wave number K with gird size $n = 64$, (b) of GMRES iterations with computational time with gird size $n = 64$

In Table 1, the results by GMRES method show that to solve the two-dimensional Helmholtz equation, when wave numbers are increases the matrix size also increases so it shows that the GMRES method needs more time to solve it also as matrix size is increases the number of iterations is also increased. For better presentation, same are illustrated in Figure 1 (a) and (b).

Deflation Results with Exact Eigenvectors

Table 2: Iteration Comparison between GMRES and deflated GMRES for different wavenumbers with use of 5 eigenvectors

K	N	GMRES iterations	CPU Time in second	Deflated GMRES iterations	CPU Time in second
10	16	71	0.473830	53	0.1435
20	32	221	1.216765	177	5.8303
30	48	423	11.032648	371	57.3279
40	64	826	94.470132	693	346.8009
50	80	1256	413.04445	1117	1343.3733

Table 3: Iteration Comparison between GMRES and Deflated GMRES with different wavenumber and 10 exact eigenvectors

K	N	GMRES Iterations	CPU Time in second	Deflated GMRES iterations	CPU Time in second
10	16	71	0.435149	45	0.101087
20	32	221	1.176970	144	4.971228
30	48	423	10.613074	311	47.534826
40	64	826	94.055999	459	224.676751
50	80	1256	432.717566	669	809.241908

Table 4: Iteration Comparison between GMRES and Deflated GMRES for different wavenumber and 20 exact eigenvectors

K	N	GMRES iterations	CPU Time in second	Deflated GMRES iterations	CPU Time in second
10	16	71	0.085043	24	0.082999
20	32	221	1.234772	104	3.603809
30	48	423	11.211972	215	32.051189
40	64	826	96.056060	253	121.038439
50	80	1256	419.873603	410	475.037916

In Tables 2-4, it can be observed that as the grid size n increases, the number of iterations in GMRES also increases. In these tables, the comparison of GMRES and Deflated GMRES, it has been observed that deflated GMRES improves convergence and also reduces the number of iterations. The Figure 2 shows comparison in one glance, where GMRES iterations re compared with deflated GMRES iterations, deflated by 5, 10 and 20 exact eigenvectors. Further, convergence history of all competing solvers is depicted in Figure 3, this conclusively shows that more eigenvalues reduces norm faster as compared with no deflation or deflation with fewer eigenvectors.

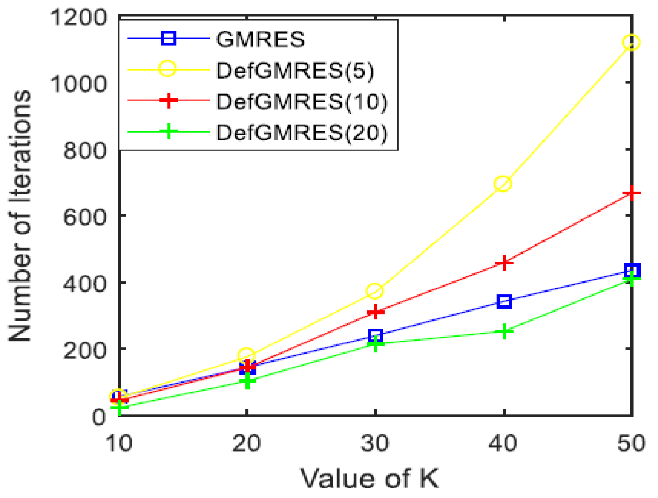

Figure 2: GMRES iterations and Deflated GMRES using 5, 10 and 20 exact Eigen vectors with different wave number

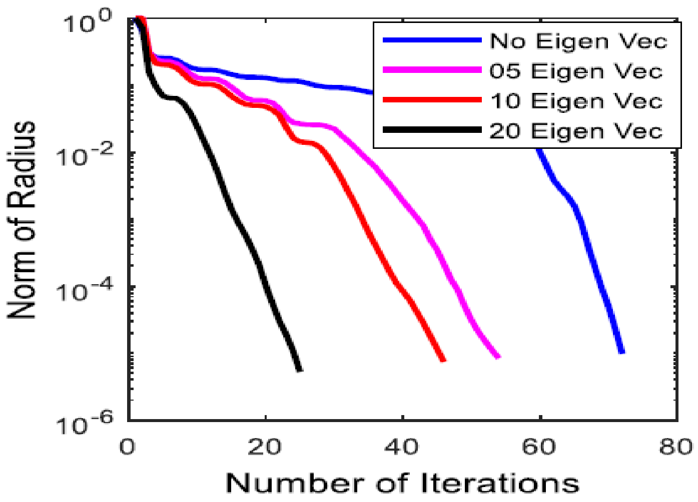

Figure 3: Convergence History; Norm of Residual at every iteration fixed k=10 with 0, 5, 10 and 20 eigenvectors

Deflation with Ritz Vectors Results

To solve problem with second right hand side recycling the Ritz vectors of

$f\ (x,y) = 5x^3 - 5x^2\ y^2 - 5xy^2 - 8x - 8y$

$$K = \frac{2\pi f}{\lambda}$$

Table 5: Iteration Comparison between GMRES and Deflated GMRES for different wavenumber with 20 eigenvalues

(for second system using Ritz vector)

K	N	GMRES Iterations	CPU Time in second	Deflated GMRES iterations	CPU Time in second
10	16	71	0.109317	89	0.187560
20	32	221	1.295302	278	4.834375
30	48	423	12.493921	507	40.785079
40	64	826	98.817696	1012	256.136442
50	80	1256	455.271950	1457	800.430126

Table 6: Iteration Comparison between GMRES and Deflated GMRES for different wavenumber with 40 eigenvalues are used
(for second system using Ritz vector).

K	N	GMRES Iterations	CPU Time (s)	Deflated GMRES Iterations	CPU time (s)
10	16	71	0.139638	68	0.138373
20	32	221	1.399449	263	3.970580
30	48	423	11.227726	505	35.851836
40	64	826	95.915184	1022	233.20257
50	80	1256	414.773581	1464	829.18840

Table 7: Iiteration Comparison between GMRES and Deflated GMRES for different wavenumber with 60 eigenvalues are used

(for second system using Ritz vector)

K	N	GMRES Iterations	CPU Time (s)	Def. GMRES Iter.	CPU Time (s)
10	16	71	0.176844	53	0.163298
20	32	221	1.385871	246	3.911741
30	48	423	11.103388	479	33.383147
40	64	826	95.581182	1003	231.582307
50	80	1256	414.773581	1443	777.706890

In Tables 5-7, it is observed that as the number of grid points n increases as number of iterations are increases in GMRES, also shown in these tables, comparison of GMRES and Deflated GMRES. Eigenvalues, and has used.

Similarly, Figure 4 compares iterations for GMRES and deflated GMRES with different number of Ritz vectors. A faire comparison has been noticed. The convergence history of methods is illustrated in Figure 5, which also confirms that more Ritz vectors are beneficial in terms of norm. Deflation with more Ritz vectors helps to reduce norm with fewer iterations. Note that deflation has been performed using 20, 40 and 60 Ritz vectors.

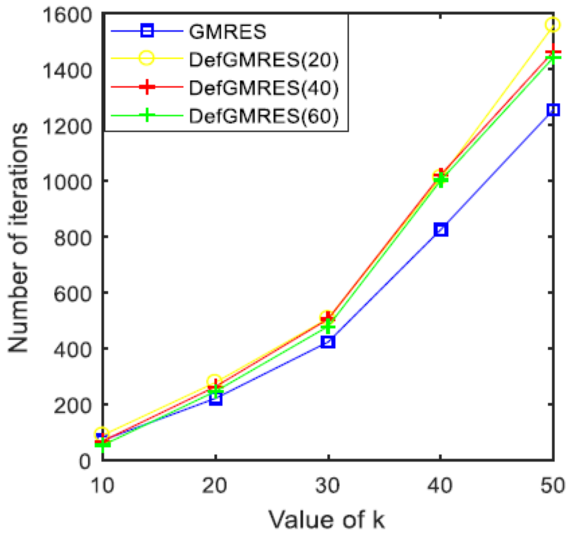

Figure 4: GMRES iteration and Deflated GMRES using 20, 40 and 60 Eigen values at different

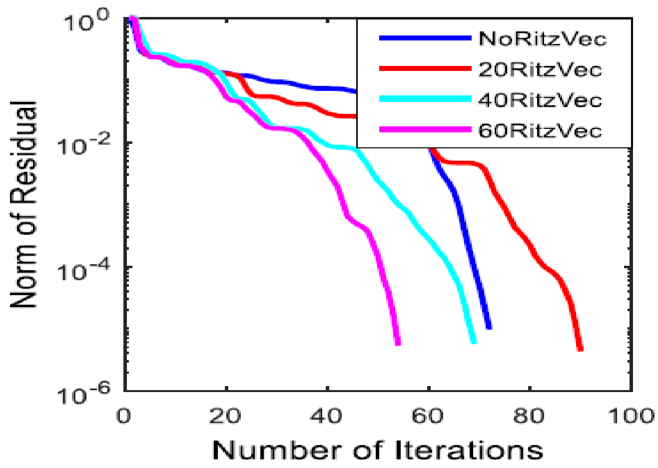

Figure 5: Convergence History; Norm of Residual at every iteration fixed k=10

Conclusion

To accelerate GMRES iterations, the method has been deflated with eigenvectors in first step and with Ritz vector in second instance. For simplicity, a linear system obtained the Helmholtz equation has been tested where two different right-hand side vectors are considered. A notable gain has been noticed in results. The results show that GMRES with 60 Ritz vectors reduce the CPU time and number of iterations it also works on residual and it reduces the residual values as compare to the GMRES method with eigenvectors and eigenvalues. and take one example of Poisson matrix for linear systems with single right-hand side vector and also use eigenvalues as deflation and it reduces the time duration, number of iterations and residuals

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