Journal of Statistics, Computing and Interdisciplinary Research
ISSN (P): 2707-7101

J. Stat. Comp. and Interdis. Rese.
Volume 4(1) (2022), 61-73
http://scir.wum.edu.pk/index.php/ojs

Delta Perturbation Method for Thin Film Flow of a Third Grade Fluid on a Vertical Moving Belt

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Abstract

In this paper, we theoretically investigate the lift problem for thin film flow of a third-grade fluid on a vertical moving belt by using of delta perturbation method (DPM). The continuity and momentum equations model the problem and DPM method is employed to solve equations analytically. The DPM is type of perturbation technique and introduced by Bender and his colleagues in 1980s. Substitutions $\beta_2 + \beta_3 = 0$and subsequently DPM method leads to Newtonian solution. The closed form expressions for velocity and temperature profiles, average velocity, volume flux and net upward flow are worked out. The relation between various emerging parameters and velocity profile v_z, and temperature profile are presented graphically and as well as by using table, from where, we have pointed out that third grade fluid will uplift quickly as the increase of dynamic viscosity and decrease of the constant parameters, density and uniform thickness, it is also noted for proposed model that temperature distribution rises for the constant parameters, uniform thickness and density of the fluid and small values of thermal conductivity and dynamic viscosity of the fluid.

Keywords: Thin film flow, Delta perturbation method, third grade fluid.

Introduction

In past years, researcher and scientist examined the phenomena for flow of thin film both experimentally and theoretically. It is extremely important to comprehend their mechanics a few applications due to the flows are universal and plentiful in nature. In the spaces including energy conversion, photovoltaic, biomedical engineering, energy efficiency, flat panel and pharmaceuticals shows have found in currant years in fast extension. Especially, in inconstancy of film covering interaction's such coatings on, lining of mammalian lungs, manufactured products, flow for surface active fluids, microchip fabrication, condensate motion, painting, microchip production, and on food dispersion of sauce are their imperative and intrigued applications (Alam et al., 2021; Bird, 1987; Khaskheli et al., 2020; A. A. Memon et al., 2019; KN Memon et al., 2014; Shah et al., 2019).

Mostly application which are used in our daily life of fluid in industries, biological sciences and any other technology have a nonlinear connection among shear stress and rate of deformation, that type of fluid are called as a non-Newtonian fluid and these types of fluids are categorized are ordered by their constitutive equations. For non-Newtonian fluids the shear stress has nonlinear function with rate of deformation and generally mathematically solution concerning flow problem for these types of fluids are overall harder to acquire for both numerically and as well as analytically (Siddiqui et al., 2013). Recently ordered as "n" fluids have acquired significant consideration, among these types of fluids, , one unique subclass is second grade fluid and which related with second-order truncation. Albeit for steady flow a second-grade fluid model displays normal stress impacts, it doesn't have the property of shear thickening or shear thinning that numerous fluids show (Zahid et al., 2017). In any case, such type of phenomena can be described by third grade fluids (Fosdick and Rajagopal, 1980). For the behavior of non-Newtonian, the third order fluid model signifies a more advance, albeit inconclusive, endeavor toward a more thorough depiction. In this work third-order fluid model will be considered because of its significance in daily life. Fosdick and Rajagopal was developed the theory concerning stability and thermodynamics of third grade fluids (Fosdick and Rajagopal 1980)., A lot of researcher successfully studied theoretically and solve nonlinear differential equations which are govern by the flow of third grade fluid, which was a challenging task (Zahid et al., 2017).

Commonly for calculating the analytical solution's, the perturbation method is employed to get solution of the nonlinear differential equation having small/large ε parameter; albeit it is not guarantee for all nonlinear differential equations due to involvement of small/large parameter. Therfore, there is solid prerequisite for to evolve new tchique for analytical methods, on these methods exploration was specified by (He, 2006).

In the last part of the 1980's, Bender and partners presented a new technique, which is called delta perturbation method. Actually, this technique is the type concerning with Perturbation method(Abraham-Shrauner et al., 1988; Bender and Milton, 1988; Bender and Rebhan, 1990; Bender et al., 1992). In this method one develops in powers of a nonlinearity which is existing into nonlinear differential equation (Van Gorder, 2011a; 2011b; Ji-huan, 2002). This theory was first applied into the problems related to theory of quantum field, in a lot of areas of science, this method found sufficient application, especially for nonlinear differential equations(see, on behalf of illustration,(Ji-huan, 2002; Van Gorder, 2011a; 2011b; Abraham-Shrauner, 1992; Bender and Jones, 1988; Bender and Milton 1988; Kamran et al., 2021). In this manuscript, study of think film flow for a third-grade fluid is presented where delta perturbation method is used. Analytical solution the resulting differential equations subject to boundary conditions, are obtained and for substituting $\beta_2 + \beta_3 = 0$, we retrieve the solution for Newtonian fluid (Munson et al., 2013). Also, expressions for velocity profile, temperature profile, flow rate, average velocity and net upward flow for lift problem are calculated. To best of knowledge of authors, the acceptable solution has not been reported in literature so far.

The manuscript is structured as follows. The next section describes the governingequations i.e. the Naiver-Stokes equation. Problem formulations is given in the Section 3. Section number 4 deals with solution of the problem by using delta perturbation method. Results and discussion are given in section number 5, while conclusion remarks are provided in section number 6.

Basic Equations

The basic equations governing the motion of an incompressible fluid, neglecting the thermal effects and body forces, are

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1}$$

$$\rho \frac{D\boldsymbol{V}}{Dt} = \rho \boldsymbol{b} + \nabla \cdot \boldsymbol{\tau} - \nabla p, \tag{2}$$

$$\rho C_p \frac{D\theta}{Dt} = \frac{1}{2} tr(\boldsymbol{\tau} \boldsymbol{A}_1) + k\nabla^2\theta \tag{3}$$

Here $\boldsymbol{V}$ represents to velocity vector, $\boldsymbol{b}$ be the body force, $\boldsymbol{\tau}$ symbolize to extra stress tensor, for dynamic pressure is symbolized as p, θ represent to the temperature distribution of the fluid, C_p is specified for specific heat, k symbolized for thermal conductivity and $\frac{D}{Dt}$ signifies to total derivative, which is equal to $\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla$ and $\frac{D\boldsymbol{V}}{Dt}$ term is symbolizes to the substantial acceleration involving of the convective derivative $(\nabla \cdot \boldsymbol{V})$ and the local derivative $\frac{\partial \boldsymbol{V}}{\partial t}$. The extra stress tensors defining third order fluid models is given by,

$$\boldsymbol{\tau} = \sum_{i=0}^{3} \boldsymbol{T}_i \tag{4}$$

Where

$\boldsymbol{T}_1 = \mu \boldsymbol{A}_1,$

$\boldsymbol{T}_2 = \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2,$

$\boldsymbol{T}_3 = \beta_1 \boldsymbol{A}_3 + \beta_2(\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \beta_3(tr\boldsymbol{A}_2)\boldsymbol{A}_1,$

where, μ is the coefficient of viscosity and α_1, α_2, β_1, β_2 and β_3 are material constant. The Rivilin-Ericksen tensor, $\boldsymbol{A}_n$, are defined by $\boldsymbol{A}_0 = I$, the identity tensor and

$$\boldsymbol{A}_n = \frac{D\boldsymbol{A}_{n-1}}{Dt} + (\nabla \boldsymbol{V})^T \boldsymbol{A}_{n-1} + \boldsymbol{A}_{n-1}(\nabla \boldsymbol{V}). 1 \leq n \tag{5}$$

Problem Formulation

Consider a third-grade fluid in a container. A wide moving belt passes through this container (Figure 1). This belt moves vertically upward with constant velocity U. As the belt moves upward and passes through the fluid, it picks up a film of thickness h .Gravity tends to make the fluid film drain down the belt. We assume, for simplicity that the flow is steady, laminar and uniform. We choose x-axis parallel to the fluid and normal to the belt, y-axis upward along the belt and z-axis normal to the xy-plane. As the only velocity component is in the y-direction, we take the velocity field, stress tensor and energy distribution of the form

$$\boldsymbol{V} = [u, v, w] = [0, v(x), 0], \boldsymbol{\tau} = \boldsymbol{\tau}(x), \theta = \theta(x) \tag{6}$$

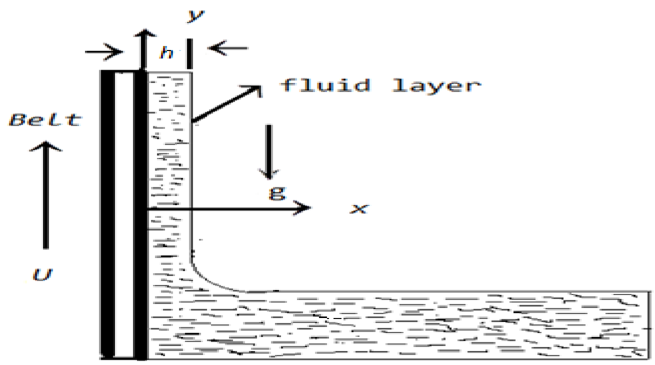

Fig. 1. Geometry of the flow of a vertical moving belt through third order fluid.

By using (6) the continuity equation (1) is satisfied identically and from (4) and (5) we obtain the components of the momentum equation (2) in the form

$$x - component\ of\ momentum: \quad (\alpha_2 + 2\alpha_1)\frac{d}{dx}\left(\frac{dv}{dx}\right)^2 = \frac{\partial p}{\partial x}, \tag{7}$$

$$y - component\ of\ momentum: \quad (\beta_2 + \beta_3)\frac{d}{dx}\left(\frac{dv}{dx}\right)^3 + \mu\frac{d}{dx}\left(\frac{dv}{dx}\right) - \rho g = \frac{\partial p}{\partial y}, \tag{8}$$

$$z - component\ of\ momentum: \quad \frac{\partial p}{\partial z} = 0. \tag{9}$$

Equation (8) shows that the pressure p does not depend on z and therefore it is a function of x and y only. Introducing the generalized pressure $\hat{p}$ by the relation

$$\hat{p} = p - (\alpha_2 + 2\alpha_1)\frac{d}{dx}\left(\frac{dv}{dx}\right)^2 \tag{10}$$

After using (10), then equations (7) and (8) will take the form as:

$$\frac{\partial \hat{p}}{\partial x} = 0 \tag{11}$$

$$6(\beta_2 + \beta_3)\frac{d^2v}{dx^2}\left(\frac{dv}{dx}\right)^2 + \mu\frac{d^2v}{dx^2} - \rho g = \frac{\partial \hat{p}}{\partial y} \tag{12}$$

Equation (11) shows that $\hat{p}$ is independent of the axial distance x, and since the velocity on left-hand side of the above equation is a function of x while the pressure$\hat{p}$on the right- hand side is a function of y alone, the two sides can be equal only if each is constant. As there is no pressure gradient along y-direction, this constant can be taken to be zero so that

$$6(\beta_2 + \beta_3)\frac{d^2v}{dx^2}\left(\frac{dv}{dx}\right)^2 + \mu\frac{d^2v}{dx^2} - \rho g = 0 \tag{13}$$

Using profile (6) in the energy equation (3), we obtain

$$k\frac{d^2\Theta}{dx^2}+\left(\mu\frac{dv}{dx}+2(\beta_2+\beta_3)\left(\frac{dv}{dx}\right)^3\right)\frac{dv}{dx}=0 \quad (14)$$

The related boundary conditions are

$$at \quad x=h, \quad \frac{\partial v}{dx}=0, \frac{\partial\Theta}{dx}=0 \quad (15)$$

$$at \quad x=0, \quad v=U, \Theta=\Theta_0 \quad (16)$$

By intigrating equation (13) then by using free space boundry condition, we have

$$\frac{dv}{dx}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv}{dx}\right)^3=\frac{\rho g}{\mu}(x-h) \quad (17)$$

Which is non-linear differential equation, from here we can use delta perturbation.

Solution of the Problem by Using Delta Perturbation Method

This method is the type of Perturbation technique and introduced by Bender and colleagues in the late 1980's**.** In this method one develops in powers of a nonlinearity which is existing into nonlinear differential equation (Gorder, 2011a; Bender and Jones, 1988; Huan, 2002; Bender and Milton, 1988; Van Gorder, 2011b; Bender et al., 1992). This theory was first applied into the problems related to theory of quantum field, in a lot of areas of science, this method found sufficient application, especially for nonlinear differential equations. To solve equation (17) approximately using the δ - *expansion method*, we let $3=1+\delta$ and solve

$$\frac{dv}{dx}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv}{dx}\right)^{1+\delta}=\frac{\rho g}{\mu}(x-h) \quad (18)$$

To solve equation (18) by using delta perturbation, we seek a solution $v(x)$in the form of series in powers of δ, which is:

$$v(x)=v_0(x)+\delta v_1(x)+\delta^2 v_2(x)+\ldots \quad (19)$$

Substituting equation (19) in equation (18) and (16), we get the following problems of different order's with corresponding condition is

Zeroth order problem

$$\delta^0: \frac{dv_0}{dx}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv_0}{dx}\right)=\frac{\rho g}{\mu}(x-h), with v_0=U, \quad at\ x=0 \quad (20)$$

First order problem

$$\delta^1: \frac{dv_1}{dx}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv_1}{dx}+\frac{dv_0}{dx}ln\left(\frac{dv_0}{dx}\right)\right)=0, with v_1=0, \quad at\ x=0 \quad (21)$$

Second order problem

$$\delta^2: \frac{dv_2}{dx}+2\left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{dv_2}{dx}+\frac{dv_1}{dx}ln\left(\frac{dv_0}{dx}\right)+\frac{1}{2}\left(\frac{dv_0}{dx}\right)\left(ln\frac{dv_0}{dx}\right)^2\right)=0, with v_2=0, \quad at\ x=0 \quad (22)$$

Solution of zero order, first order, second order problem with related condition is,

$$\boldsymbol{v_0=U+\frac{\rho g x(-2h+x)}{2(\mu+2\beta_2+2\beta_3)}} \quad (23)$$

$$v_1=\frac{g(\beta_2+\beta_3)\rho\left\{-2h^2\,ln\left(1-\frac{x}{h}\right)+(2h-x)x\left(-1+2\,ln\left(\frac{\rho g(-h+x)}{\mu+2\beta_2+2\beta_3}\right)\right)\right\}}{2(\mu+2\beta_2+2\beta_3)^2} \quad (24)$$

$v_2=$

$$\frac{-g\rho(\mu-2\beta_2-2\beta_3)(\beta_2+\beta_3)}{4(\mu+2\beta_2+2\beta_3)^3}\left\{\begin{array}{c} x(-2h+x)+2(h-x)^2\left(-1+ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\right)ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right) \\ +2h^2\,ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)-2h^2\,ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2 \end{array}\right\} \quad (25)$$

The series solution up to the second order, after Substituting the solutions of $v_0, v_1 and v_2$, once we obtain by ignoring third and higher order solution, we obtain

$$v(x)=U+\frac{\rho g x(-2h+x)}{2(\mu+2\beta_2+2\beta_3)} + \frac{\delta g(\beta_2+\beta_3)\rho\left\{-2h^2\,ln\left(1-\frac{x}{h}\right)+(2h-x)x\left(-1+2\,ln\left(\frac{\rho g(-h+x)}{\mu+2\beta_2+2\beta_3}\right)\right)\right\}}{2(\mu+2\beta_2+2\beta_3)^2}$$
$$+\frac{-g\rho\delta^2(\mu-2\beta_2-2\beta_3)(\beta_2+\beta_3)}{4(\mu+2\beta_2+2\beta_3)^3}\left\{\begin{array}{c} x(-2h+x)+2(h-x)^2\left(-1+ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\right) \\ ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right) \\ +2h^2\,ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)-2h^2\,ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2 \end{array}\right\} \quad (26)$$

By using equation (17), the energy equation can be written as

$$\frac{d^2\Theta}{dx^2}=-\frac{\rho g}{k}(x-h)\frac{dv}{dx} \quad (27)$$

Solution for equation (27 by using equation (15) and (16), we obtain:

$$\Theta=\frac{1}{864k(\mu+2\beta_2+2\beta_3)^3}[72\mu^2\left(12k\Theta_0\mu-g^2x\left(-4h^3+6h^2x-4hx^2+x^3\right)\rho^2\right)$$
$$+(\beta_2+\beta_3)\{\mu(5184k\Theta_0\mu+g^2x\left(-4h^3+6h^2x-4hx^2+x^3\right)\left(-288+\delta(-84+37\delta)\right)\rho^2$$
$$+12g^2\delta\rho^2(h^4(-12+7\delta)\,ln\left(-\frac{gh\rho}{\mu+2\beta_2+2\beta_3}\right)^2-6h^4\delta\,ln\left(-\frac{gh\rho}{\mu+2\beta_2+2\beta_3}\right)^2$$
$$+(h-x)^4\,ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\left(12-7\delta+6\delta\,ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\right)))$$
$$+2(\beta_2+\beta_3)(5184k\Theta_0\mu-g^2x\left(-4h^3+6h^2x-4hx^2+x^3\right)\left(144+\delta(84+37\delta)\right)\rho^2$$
$$+12g^2\delta\rho^2(-h^4(12+7\delta)\,ln\left(-\frac{gh\rho}{\mu+2\beta_2+2\beta_3}\right)+6h^4\delta\,ln\left(-\frac{gh\rho}{\mu+2\beta_2+2\beta_3}\right)^2$$
$$+(h-x)^4\,ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\left(12+7\delta-6\delta\,ln\left(\frac{g(-h+x)\rho}{\mu+2\beta_2+2\beta_3}\right)\right))+3456k\Theta_0(\beta_2+\beta_3))\}] \quad (28)$$

Flow Rate and Average Film Velocity

To determine the flow rate Qper unit width, we use formula

$$Q=\int_0^h v(x)dx$$

By making use of (26) in the above formula we obtain

$$Q = hU + \frac{1}{54(\mu + 2\beta_2 + 2\beta_3)^3} h^3 [-18g\mu^2\rho + (\beta_2 + \beta_3)\{\mu(4g(-18 + \delta^2)\rho$$
$$-12\delta\rho g + 3\delta((5g\delta\rho + 12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu + 2(\beta_2 + \beta_3)}\right)\right)$$
$$-9g\delta\rho\, ln\left(-\frac{gh\rho}{\mu + 2(\beta_2 + \beta_3)}\right) + 6g\delta\rho\, ln\left(-\frac{gh\rho}{\mu + 2(\beta_2 + \beta_3)}\right)^2))$$
$$+2(2g(-18 + 25\delta^2)\rho - 12\delta\rho g + 3\delta((103g\delta\rho + 12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu + 2(\beta_2 + \beta_3)}\right)\right)$$
$$-135g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right) + 30g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2))(\beta_2 + \beta_3)\}] \quad (29)$$

The average film velocity $\bar{V}$ is then given by $\bar{V} = Q/h$ which in view of (29), yields

$$\overline{V} = U + \frac{1}{54(\mu+2\beta_2+2\beta_3)^3} h^2 [-18g\mu^2\rho + (\beta_2 + \beta_3)\{\mu\left(4g(-18 + \delta^2)\rho - 12\delta\rho g\ +\right.$$
$$3\delta\left((5g\delta\rho + 12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu+2(\beta_2+\beta_3)}\right)\right) - 9g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right) +\right.$$
$$\left.\left.6g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2\right)\right) + 2(2g(-18 + 25\delta^2)\rho - 12\delta\rho g\ + 3\delta((103g\delta\rho +$$
$$12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu+2(\beta_2+\beta_3)}\right)\right) - 135g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right) +$$
$$30g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2))(\beta_2 + \beta_3)\}] \quad (30)$$

For net upward flow of fluid if $\overline{V} > 0$ which implies that

$$U > \frac{-1}{54(\mu+2\beta_2+2\beta_3)^3} h^2 [-18g\mu^2\rho + (\beta_2 + \beta_3)\{\mu\left(4g(-18 + \delta^2)\rho\ -\right.$$
$$12\delta\rho g\ 3\delta\left((5g\delta\rho + 12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu+2(\beta_2+\beta_3)}\right)\right) - 9g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right) +\right.$$
$$\left.\left.6g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2\right)\right) + 2(2g(-18 + 25\delta^2)\rho - 12\delta\rho g\ 3\delta((103g\delta\rho +$$
$$12\rho g)\left(ln(-h) + ln\left(\frac{g\rho}{\mu+2(\beta_2+\beta_3)}\right)\right) - 135g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right) +$$
$$0g\delta\rho\, ln\left(-\frac{gh\rho}{\mu+2(\beta_2+\beta_3)}\right)^2))(\beta_2 + \beta_3)\}] \quad (31)$$

Remarks: Here we have pointed out that for taking$\beta_2 + \beta_3 = 0$, the results will be reviewedfor the Newtonian fluid solution, which is given in [19].

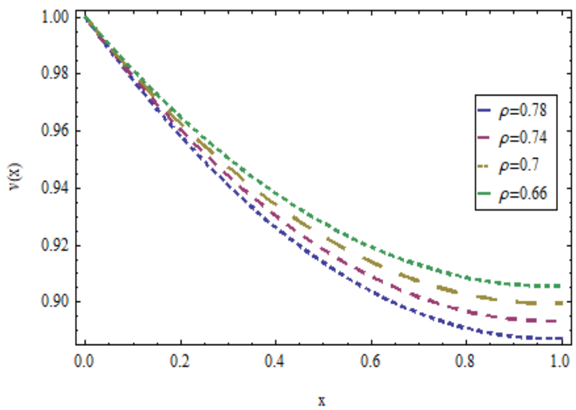

Figure 2: Effect of ρon velocity profile, when $h = 1cm, U = \frac{1cm}{s}, \beta_2 = 0.1, \beta_3 = 0.1, \mu = 38.5poise$

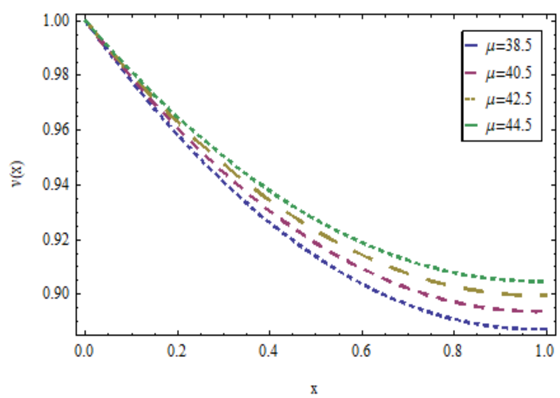

Figure 3: Effect of μon velocity profile, when $h = 1cm, U = 1cm/s, \rho = 0.78g/cm^3, \beta_2 = 0.1, \beta_3 = 0.1$

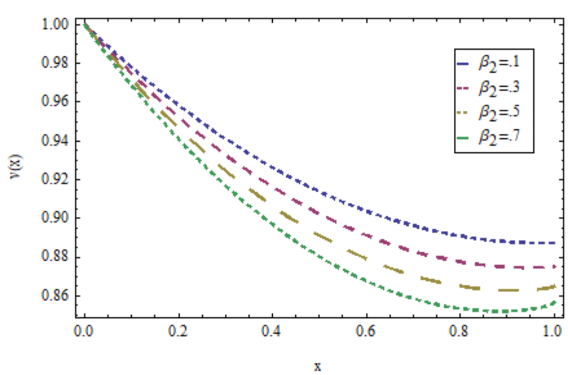

Figure 4: Effect of $\beta_2$on velocityprofile,when $h = 1cm, U = 1cm/s, \rho = 0.78g/cm^3, \beta_3 = 0.1, \mu = 38.5$poise

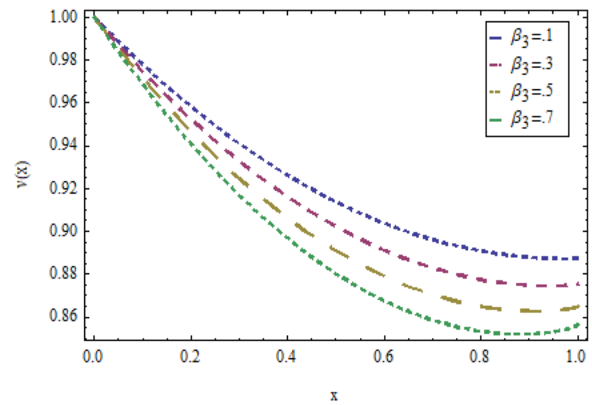

Figure 5: Effect of $\beta_3$on velocity profile, when $h = 1cm, U = \frac{1cm}{s}, \rho = \frac{0.78g}{cm^3}, \beta_2 = 0.1, \mu = 38.5poise$

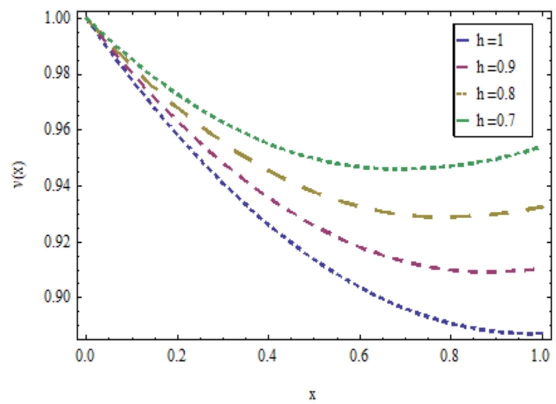

Figure 6: Effect of $h$on velocity profile, when $U = 1cm/s, \rho = 0.78g/cm^3, \beta_2 = 0.1, \beta_3 = 0.1, \mu = 38.5poise$

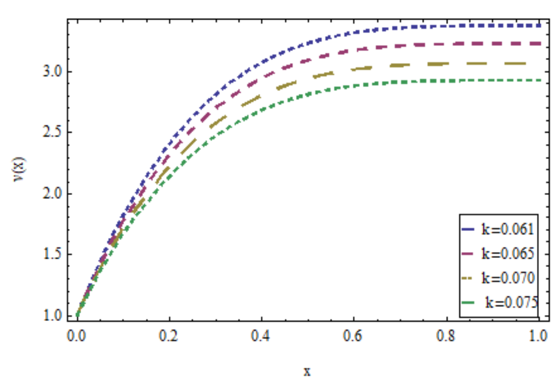

Figure 7: Effect of $k$on Temperature profile, when$\Theta = 1, \rho = 0.78g/cm^3, \delta = 2, \beta_2 = 0.1,\ \beta_3 = 0.1, \mu = 38.5poise$

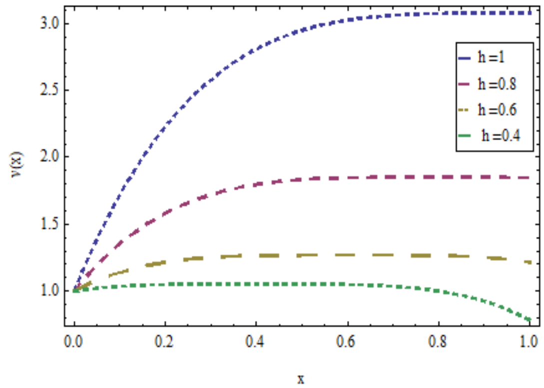

Figure 8: Effect of h on temp. profile when $\Theta = 1, k = 0.061\ W/cm.K,\ \beta_3 = 0.1, \rho = 0.78\ g/cm^3, \delta = 2, \beta_2 = 0.1, \mu = 38.5\ poise$

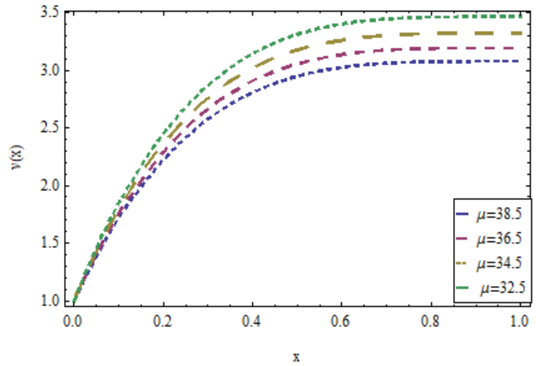

Figure 9: Effect of μontemp. profil when $\Theta = 1, k = 0.061\frac{W}{cm}.k, \rho = 0.78\ g/cm$

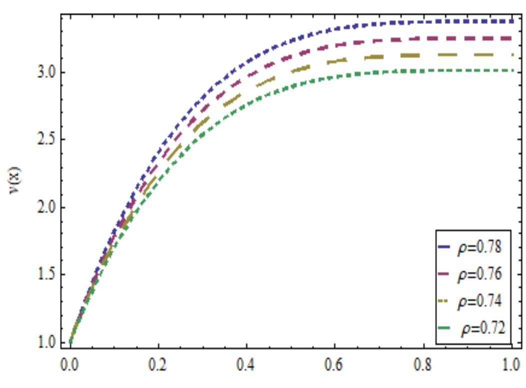

Figure 10: Effect of ρ on temp. profile, when $\Theta = 1, k = 0.061W/cm.K, \mu = 38.5poise, \delta = 2, \beta_2 = 0.1, \beta_3 = 0.1, h = 1cm$

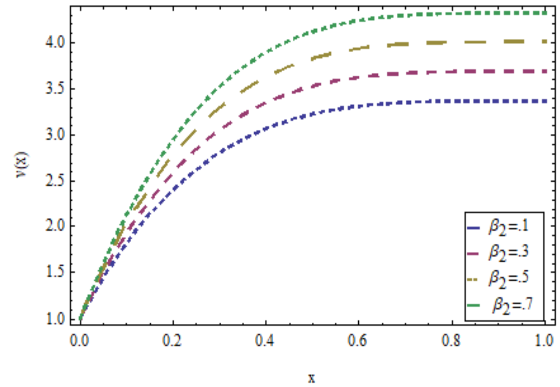

Figure 11: Effect of $\beta_2$on temp. profile,when$\Theta = 1, k = \frac{0.061W}{cm}.K, h = 1cm, \mu = 38.5poise, \delta = 2, \rho = \frac{0.78g}{cm^3}, \beta_3 = 0.1$

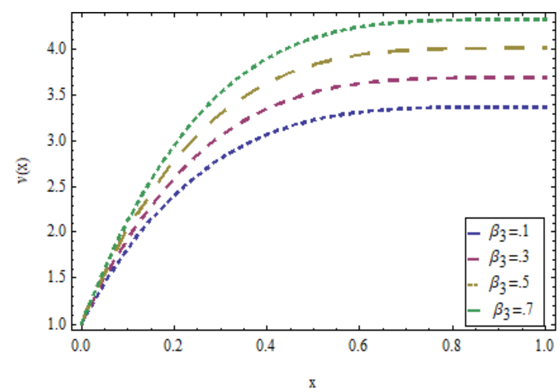

Figure 12: Effect of $\beta_3$on temp. profile, when $\Theta = 1, k = \frac{0.061W}{cm}.K, h = 1cm, \mu = 38.5poise, \delta = 2, \rho = \frac{0.78g}{cm^3}, \beta_2 = 0.1.$

Table 1: Effect of β_3 on Velocity Profile when $h = 1cm, U = \frac{1cm}{s}, \rho = \frac{0.78g}{cm^3}, \beta_2 = 0.1, \mu = 38.5poise$

x	$\beta_3 = 0.1$	$\beta_3 = 0.3$	$\beta_3 = 0.5$	$\beta_3 = 0.7$
0.	1.	1.	1.	1.
0.2	0.958265	0.955298	0.952356	0.94944
0.4	0.926197	0.921127	0.916109	0.91115
0.6	0.903759	0.89744	0.891212	0.885091
0.8	0.890859	0.884118	0.877536	0.87114
1.	0.884409	0.877457	0.870698	0.8641645

Table 2: Effect ofβ_3 on Temperature Profile$when \Theta = 1, k = \frac{0.061W}{cm}.K, h = 1cm, \mu = 38.5poise, \delta = 2, \rho = 0.78g/cm^3, \beta_2 = 0.1.$

x	$k = 0.061$	$k = 0.065$	$k = 0.070$	$k = 0.075$
0.	1.	1.	1.	1.
0.2	2.40933	2.32231	2.22754	2.14543
0.4	3.07302	2.94502	2.80561	2.68481
0.6	3.31552	3.17253	3.0168	2.88186
0.8	3.36976	3.22341	3.06402	2.92591
1.	3.42433	3.27429	3.11124	2.96996

Table 3: Effect of k on Temperature Profile $when \Theta = 1, \mu = 38.5poise, \rho = 0.78, \delta = 2, \beta_2 = 0.1, \beta_3 = 0.1, h = 1cm.$

x	$\beta_3 = 0.1$	$\beta_3 = 0.3$	$\beta_3 = 0.5$	$\beta_3 = 0.7$
0.	1.	1.	1.	1.
0.25	2.63104	2.84941	3.06437	3.27114
0.5	3.23013	3.5303	3.82963	4.11955
0.75	3.3647	3.68199	4.00022	4.30946
1.	3.49929	3.83368	4.17081	4.49951

Results and Discussion

In the above sections, we studied lift problem for thin film flow for third grade fluid using an delta perturbation method, which is type of perturbation technique and introduced by Bender and his colleagues in 1980's. The fluid is characterized as steady, uniform, isothermal and incompressible, which leads to nonlinear ordinary differential equation. Analytical solutions for the second order differential equation is obtained which gives velocity profile of fluid and temperature distribution. The variation of velocity profile $v_z$and temperature profile has been investigated on different parameters. The effects of the dynamic viscosityμ, uniform thickness$h$, densityρandconstant parameters $\beta_2$and$\beta_3$on velocity profile are observed through throgh figures (2) - (6) as well as Table 1 and effect of thermal conductivity$k$ and other parameters such as $\mu, h, \rho,$ $\beta_2$and$\beta_3$are observed for temperature profile in figures (7) - (12) also in Table 2-3. From figures (2) - (6) it is detected that the magnitude of velocity increase with the increase dynamic viscosityμ and decreases for the increase of constant parameters $\beta_2$andβ_3, density$\rho$and uniform thickness$h$. In figures (7) - (12) it is noticed that the magnitude of temperature distribution losses with the increases of dynamic viscosity$\mu$and thermal conductivity$k$ and rises for the increase of densityρ, uniform thickness$h$, andconstant parameters $\beta_2$andβ_3. From Table 1-2, it is detected that with the rises of constant parameters $\beta_3$the magnitude of velocity field decreases and magnitude of temperature profile increases. From Table 3 it can notices that magnitude of temperature rises for the decrease of thermal conductivity$k$.

Conclusions

Considering equation for steady, uniform, isothermal and incompressible thin film flow on behalf ofthird order fluid on a vertical belt for lift problem, we have obtained analytic solutions for non-linear ordinary differential equation by using delta perturbation method for velocity field and temperature distribution, which is type of perturbation technique and introduced by Bender and his colleagues in 1980's. After using the value velocity field, we have successively found out the flow rate, average velocity and also net upward flow. Also, we have exactly retrieved the Newtonian solution for setting the fluid parameter$\beta_2 + \beta_3 = 0$.It is, noted that as the third grade fluid for this proposed problem will uplift quickly as the increase of dynamic viscosity and decrease of the constant parameters, density and uniform thickness, it is also noted for proposed model that temperature distribution rises for the constant parameters, uniform thickness and density of the fluid and small values of thermal conductivity and dynamic viscosity of the fluid

Aknowlwdgement

Authors are thankful to editorial team for their support.

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