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Exploring the Exponentiated Transmuted Inverse Rayleigh Distribution (ETIRD) in Classical and Bayesian Paradigms

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Abstract

We derived, a new three parameters continuous probability distribution called Exponentiated Transmuted Inverse Rayleigh Distribution (ETIRD). Various mathematical properties of the new distribution including mean, rth moments, moment generating function, quantile function etc. are derived. In the Classical paradigm, the estimators of the distribution are obtained using the maximum likelihood method. The Bayes estimators are derived under square error loss function (SELF) using non-informative and informative priors via the Lindley approximation technique. Bayes Estimators are compared with their corresponding maximum likelihood Estimators (MLEs) using a Monte Carlo Simulation Study under different sample sizes, different values of true parameters, using informative and non-informative priors. Performance of Bayes estimators and classical estimates is judged for the four real life data sets. The results of simulation study and real-life example show that the Bayes estimators provided better results than MLEs.

Keywords: *Inverse Rayleigh Distribution, Exponentiated method, Transmuted method, maximum likelihood estimation, Bayesian estimation, simulation study.*

1. Introduction

The probability distributions are used to defined a real-world phenomenon into mathematical function. Large number of these distributions and their generalizations are derived and the process is continuous due to complexity, diversity, and variations in theses phenomena. (Shaw & Buckley, 2007) developed a new family of distributions by adding a new parameter to an existing distribution, called transmuted – G family. The popular transmuted distributions are, the transmuted extreme valued distribution by (Aryal & Tsokos, 2009), the transmuted Lindley distribution by (Merovci, 2013), the transmuted exponentiated exponential distribution by (Merovci et al., 2017), the transmuted generalized Rayleigh distribution by (Merovci, 2014a), the transmuted Pareto distribution by (Merovci, 2014b), the Beta transmuted – G by (Yousof et al., 2015), the generalized transmuted – G by (Nofal et al. 2017), the Kumaraswamy transmuted – G family of distribution by (Afifya et al., 2016), the Exponentiated inverse Rayleigh (EIR) distribution by (ul Haq, 2016), the transmutation of four parameters generalized log-logistic distribution by (Adeyinka & Olapade, 2019).

(Gupta & Kundu, 2001a) derived the exponentiated distribution which used to develop a distribution named as exponentiated exponential distribution. This distribution has been further studied by (Nadarajah and Kotz, 2003).

(Ristić & Balakrishnan, 2012) discuss the gamma exponentiated exponential distribution, which include the lower record of exponentiated distribution. (Alzaatreh et al., 2013) generalize the new family of exponentiated TX distribution and define its properties and its special case. (Cordeiro et al., 2014) purposed the exponentiated half-logestic family of distribution and introduce the two bivariate extensions of this family. (Maxwell et al., 2019) introduce the theoretical analysis of the odd generalized exponentiated inverse Lomax distribution.

(Merovci et al., 2017) introduced a new family of continuous distribution named as the exponentiated transmuted – G (ET-G) family, which was the extension of the transmuted – G family (Shaw & Buckley, 2007).

In this article, we derived a new distribution named as the Exponentiated Transmuted Inverse Rayleigh Distribution (ETIRD), using the ET-G family of distribution proposed by (Merovci et al., 2017b). The rest of paper is organized as follows;

Section 2 presents the derivation of ETIRD along with shape of its pdf and cdf plots. In Section 3 Statistical properties like mean, variance, rth moments, moment generating function, negative moment, quantile function, skewness and kurtosis are investigated. In section 4 reliability analysis is studied. Section 5 the MLEs of the distribution are derived. In Section 6 and 7 Bayes estimators under informative and non-informative priors are derived. In Section 8 simulation study is carried out to examine performance of the MLEs and Bayesian estimators of the parameters of ETIRD. In Section 9, four real-life data sets are considered to examine the application of ETIRD in real life. Finally, the study is concluded in Section 10.

2. Exponentiated Transmuted Inverse Rayleigh Distribution

(Merovci et al., 2017) combined the exponentiated and transmuted distributions and proposed a new method named as exponentiated transmuted -G family. The cdf and pdf are defined as;

$$F(x;\alpha,\lambda,\phi) = G(x;\phi)^{\alpha-1}[1+\lambda\overline{G}\,(x;\phi)]^{\alpha} \quad (1)$$

$$f(x;\alpha,\lambda,\phi) = \alpha G(x;\phi)^{\alpha-1}.\,g(x;\phi) \quad (2)$$

By using Binomial expansion,

$$[1+\lambda\overline{G}\,(x;\phi)]^{\alpha}=\sum_{i=0}^{\alpha}\binom{\alpha}{i}[\lambda\,\overline{G}\,(x;\phi)]^{i}=\sum_{i=0}^{\alpha}\binom{\alpha}{i}\lambda^{i}\binom{\alpha}{i}\left[\sum_{k=0}^{\infty}(-1)^{k}\binom{i}{k}G(x;\phi)\right]^{k}$$

Using this expression the cdf and pdf given in equations (1) and (2) can be written as;

$$F(x;\alpha,\lambda,\phi)=G(x;\phi)^{\alpha}\textstyle\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha}\lambda^{i}\binom{\alpha}{i}\binom{i}{k}[G(x;\phi)]^{k}$$

$$F(x;\alpha,\lambda,\phi)=\textstyle\sum_{k=0}^{\infty}b\;G(x;\phi)^{\alpha+k} \quad (3)$$

$$f(x;\alpha,\lambda,\phi)=\textstyle\sum_{k=0}^{\infty}b\;(\alpha+k)G(x;\phi)^{\alpha+k-1}\,g(x;\phi) \quad (4)$$

2.1 The pdf and cdf of the ETIRD

In this study, we take the inverse Rayleigh distribution (IRD) as a base line distribution. The cdf of IRD is

$$G(x;\theta)=exp\left(-\frac{\theta}{x^{2}}\right);\qquad 0<x<\infty.$$

The pdf of IRD is written as,

$$g(x;\theta)=\frac{2\theta}{x^{3}}\;exp\left(-\frac{\theta}{x^{2}}\right);\qquad 0<x<\infty,$$

Then the cdf and pdf of ETIRD are;

$$R(x;\alpha,\lambda,\theta)=b\left[e^{-\frac{\theta}{x^{2}}}\right]^{(\alpha+k)} \quad (5)$$

$$r(x;\alpha,\lambda,\theta)=\sum_{k=0}^{\infty}b\;(\alpha+k)2\theta\;\frac{1}{x^{3}}\left[e^{-\frac{\theta}{x^{2}}}\right]^{(\alpha+k)}\quad x\geq 0,\theta>0,-1\leq\lambda\leq 1 \quad (6)$$

Where, $b=\sum_{i=0}^{\alpha}\binom{\alpha}{i}\binom{i}{k}\;(-1)^{k}\lambda^{k}$, and α is shape parameter, θ is scale parameter, and λ is a transmuted parameter.

2.3 The Graphical representations of the ETIRD

The pdf and cdf plots of ETIRD under different values $\theta,\lambda,$ and $\alpha,$ are given below.

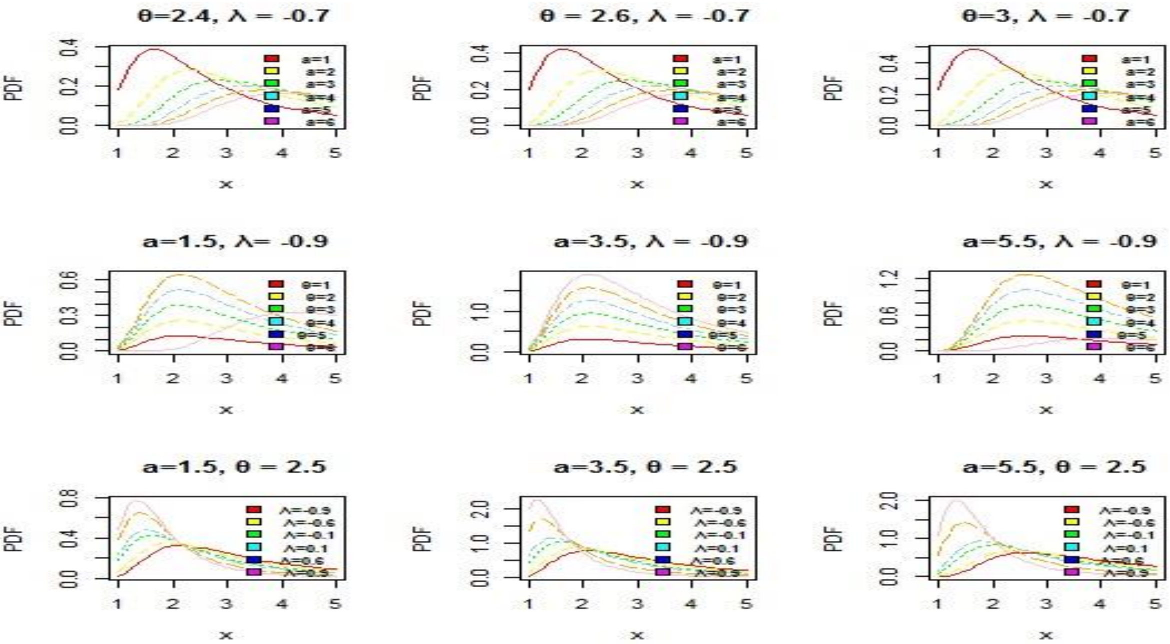

Figure 1: pdf plots of ETIRD for various values of parameters.

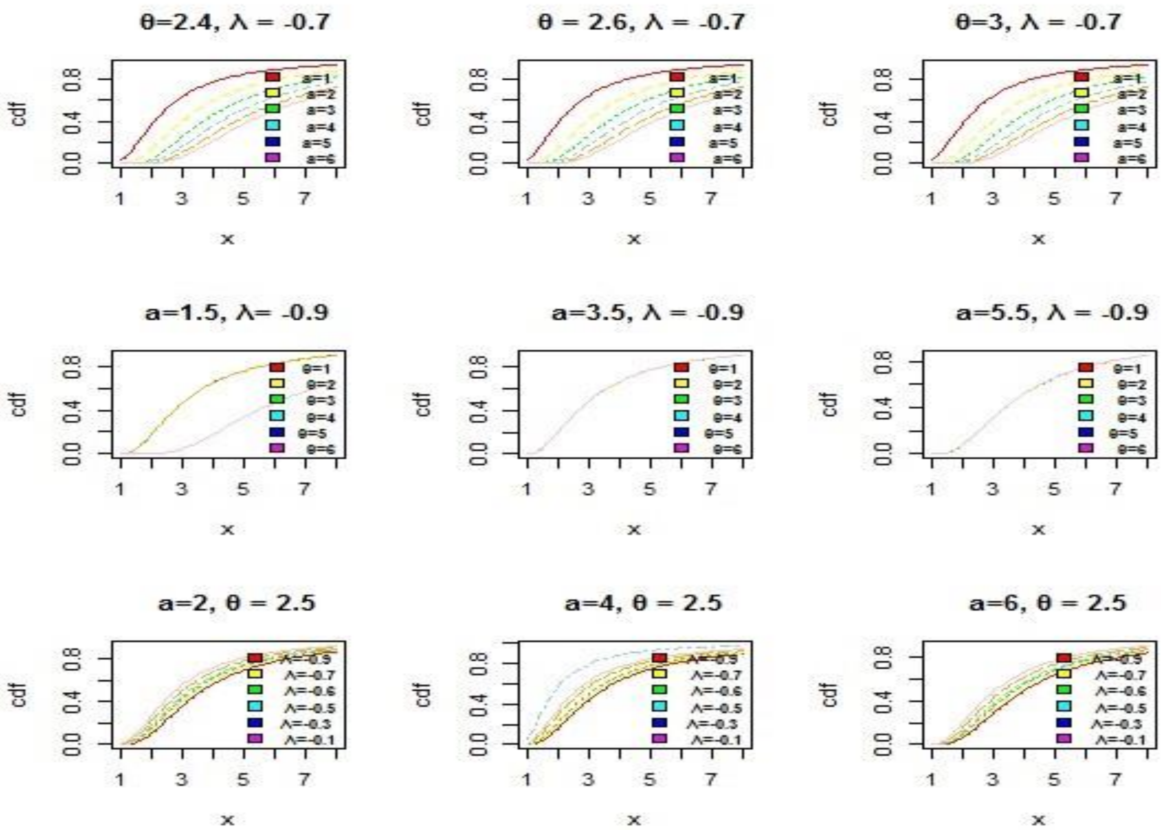

Figure 2: cdf plot of ETIRD for various values of parameters.

3. Statistical Properties

In this Section, we present some properties of the ETIRD.

3.1 Mean of the ETIRD

$$E(x) = \int_0^\infty x\, f(x)\, dx.$$

$$E(x) = \sum_{k=0}^{\infty} b\, \sqrt{\theta(\alpha + k)}\ \ \Gamma\sqrt{\pi}.$$

3.2 r^{th} Moments of ETIRD

the r^{th}moment of $E(X^r)$ is,

$$E(X^r) = \int_0^\infty x^r\, f(x)\ dx$$

$$E(X^r) = \sum_{k=0}^{\infty} b\, (\alpha + k)\theta^{\frac{r}{2}}\ \ \Gamma\left(-\frac{r}{2} + 1\right).$$

For,

$r\ = 1$ then, $\quad E(X^1) = \sum_{k=0}^{\infty} b\, \sqrt{\theta(\alpha + k)}\sqrt{\pi} = \mu_1'.$

$r\ = 2$ then, $\quad E(X^2) = \sum_{k=0}^{\infty} b\, (\alpha + k)^{\frac{2}{2}}\theta^{\frac{2}{2}}\ \ \Gamma\left(-\frac{2}{2} + 1\right) = \mu_2'\ \ does\, not\, exist.$

$r\ = 3$ then, $\quad E(X^3) = -3.5449 \sum_{k=0}^{\infty} b\, (\alpha + k)^{\frac{3}{2}}\theta^{\frac{3}{2}}\ = \mu_3'.$

$r\ = 4$ then, $\quad E(X^4) = \mu_4'\ \ does\, not\, exist.$

So, all even order non-central moments does not exist.

3.3 Moment Generating Function of ETIRD

The expression of moment generating function (m.g.f) is,

$$M_0(t) = E(e^{tx}) = \sum_{j=0}^{\infty} \frac{(t)^j}{j!} \sum_{k=0}^{\infty} b\, \theta^{\frac{j}{2}}\, (\alpha + k)^{\frac{j}{2}} \Gamma\left(-\frac{j}{2} + 1\right).$$

3.4 Characteristic Function of ETIRD

When the m.g.f does not exist for any probability distributions, we may use another important function called the characteristic function, denoted by $\phi(t)$, and defined as,

$$\phi(t) =\ E\left(e^{itx}\right) = \sum_{j=0}^{\infty} e^{itx}\ \ f(x),$$

$$\phi(t) = 1 + it\, \mu_1' + \frac{(it)^2}{2!}\, \mu_2' + \frac{(it)^3}{3!}\, \mu_3'.$$

3.5 Negative Moments of ETIRD

The negative moment generating function of rv X is defined as,

$$E(X^{-r}) = \int_0^\infty x^{-r}\, f(x)\ dx,$$

$$E(X^{-r}) = \sum_{k=0}^{\infty} b\ (\alpha+k)^{-\frac{r}{2}} \theta^{-\frac{r}{2}}\ \Gamma\left(-\frac{r}{2}+1\right).$$

3.6 Quantile Function and Quartiles

The Quantile function $Q(P)$ of the ETIRD is given below,

$$x_p = Q(P) = \left[-\frac{\theta}{\ln\left(\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda P^{\frac{1}{\alpha}}}}{2\lambda}\right)}\right]^{\frac{1}{2}}. \tag{7}$$

Particularly, the median of the distribution is derived by substituting α =1 and $P\ =\ 0.5$ in (7) is obtained as,

$$x_{0.5} = \left[-\frac{\theta}{\ln\left(\frac{(1+\lambda)-\sqrt{(1+\lambda)^2}}{2\lambda}\right)}\right]^{\frac{1}{2}}.$$

The above expression can also use for the computation of quartiles. If substitute P values as, **0.25, 0.5** and **0.75** and obtain the first, second and third quartiles.

3.7 Skewness and Kurtosis

Kurtosis measure of the peakedness of distribution. When the 3rd and 4th moments does not exist, than, use the quantile function.

$$Skewness = S = \frac{Q(6/8)-2Q(4/8)+Q(2/8)}{Q(6/8)-Q(2/8)}.$$

$$Kurtosis = K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

4 Reliability Analysis of ETIRD

In this Section, the survival function $S(x)$, hazard function $h(x)$, cumulative hazard function $ch(x)$, the reverse hazard function $rh(X)$ and Mills ratio (MR) to assert the ETIRD $(\alpha, \lambda, \theta)$ are derived.

4.1 Survival function of ETIRD

The reliability function is also known as the survival function which defined as the complementary of cdf. The Survival function is written as,

$$S(x) = 1 - F(x) = 1 - \sum_{k=0}^{\infty} b \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}.$$

4.2 Hazard function of ETIRD

The hazard function also known as the hazard rate, is derived by taking the ratio of pdf $f(x)$ and its survival function $S(x)$.

The Hazard function defined as,

$$H(x) = \frac{f(x)}{S(x)} = \frac{\sum_{k=0}^{\infty} b\ (\alpha+k)\ 2\theta\ (1/x^3)\ \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}}{1 - \sum_{k=0}^{\infty} b \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}}.$$

4.3 Cumulative Hazard Function of ETIRD

The cumulative hazard function $ch\ (x)$ shows the general link between survivor function and mortality rate. It can be obtained by mathematically by taking the negative natural log of $S(x)$.

The $ch(x)$ is defined as,

$$ch(x) = -ln[S(x)] = -ln\left[1 - \sum_{k=0}^{\infty} b \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}\right],$$

After some simplification, we obtained,

$$ch(x) = ln \sum_{k=0}^{\infty} b \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}.$$

4.4 Reverse Hazard Function of ETIRD

Reverse hazard function is defined as the ratio of the pdf and the cdf, defined as,

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{\sum_{k=0}^{\infty} b\ (\alpha+k)\ 2\theta\ (1/x^3)\ \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}}{\sum_{k=0}^{\infty} b \left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}},$$

after simplifications,

$$\tau(x) = \frac{2\theta}{x^3} \sum_{k=0}^{\infty} (\alpha + k).$$

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4.5 Mills Ratio

The concept of Mills ratio (MR) described by (John P.Mills, 1926). It is defined as ratio of the $S(x)$ and $f(x)$. The Mills ratio of the ETIRD is given as:

$$M(x)=\frac{S(x)}{f(x)}=\frac{1-\sum_{k=0}^{\infty} b\left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}}{\sum_{k=0}^{\infty} b\;(\alpha+k)\;2\theta\;(1/x^3)\;\left[e^{-\frac{\theta}{x^2}}\right]^{(\alpha+k)}}$$

5. Maximum Likelihood Estimation (MLE)

Due to possessing the asymptotic properties of normality and efficiency, the MLE have greater importance in statistical inference.

Let $x_1, x_2, \dots, x_n$ be independent and identically distributed (i.i.d), a random sample of size n from the ETIRD,

Then the likelihood function is given;

$$L(x;\alpha,\lambda,\theta)=\alpha^n\prod_{i=1}^{n}\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}\left[1+\lambda-2\lambda\,e^{-\frac{\theta}{x^2}}\right]\left[e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}\left[1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}, \qquad (8)$$

and the log-likelihood function is defined as;

$$l=(x;\alpha,\lambda,\theta)=nln(\alpha)+\sum_{i=1}^{n}\ln\left[\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}\right]+\sum_{i=1}^{n}ln\left[1+\lambda-2\lambda\,e^{-\frac{\theta}{x^2}}\right]+$$
$$(\alpha-1)\sum_{i=1}^{n}ln\,e^{-\frac{\theta}{x^2}}+(\alpha-1)\;\sum_{i=1}^{n}ln\left[1+\lambda-\lambda\,e^{-\frac{\theta}{x^2}}\right]$$

(9)

Differentiate the equation (20) w.r.t α, λ and θ and equate to zero. The MLE of α, λ and θ can be obtained from the following non-linear equations,

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}\sum_{k=0}^{\infty}lne^{-\frac{\theta}{x^2}}+\sum_{i=1}^{n}ln\left[1+\lambda-\lambda\,e^{-\frac{\theta}{x^2}}\right]=0$$

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{\left(1-2e^{-\frac{\theta}{x^2}}\right)}{\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)}+(\alpha-1)\sum_{i=1}^{n}\frac{\left(1-e^{-\frac{\theta}{x^2}}\right)}{\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)}=0$$

$$\frac{\partial l}{\partial\theta}=\sum_{i=1}^{n}\frac{\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}\left(-\frac{1}{x^2}+e^{-\frac{\theta}{x^2}}\left(\frac{2}{x^3}\right)\right)}{\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}}-2\lambda\sum_{i=1}^{n}\frac{e^{-\frac{\theta}{x^2}}}{x^2\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{\theta}{x^2}}\left(-\frac{1}{x^2}\right)}{\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)}=0$$

These normal equations are not in closed form, the MLEs of ETIR are estimated through the Newton - Raphson method by using *maxLik* package in R language. The Fisher Information Matrix is defined as,

$$I(\alpha,\lambda,\theta)=\begin{bmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\alpha\partial\theta}\\ \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\theta}\\ \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\theta\partial\lambda} & \frac{\partial^2 l}{\partial\theta^2}\end{bmatrix},$$

where, $\frac{\partial^2 l}{\partial\alpha^2}=\frac{n}{\alpha^2}$,

$$\frac{\partial^2 l}{\partial\lambda^2}=-\sum_{i=1}^{n}\frac{\left(1-2e^{-\frac{\theta}{x^2}}\right)^2}{\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\theta^2}=\sum_{i=1}^{n}\frac{1}{\theta^2}-\sum_{i=1}^{n}\frac{2\lambda e^{-\frac{\theta}{x^2}}(1+\lambda)}{x^4\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^2}-(\alpha-1)\sum_{i=1}^{n}\frac{\lambda e^{-\frac{\theta}{x^2}}(1+\lambda)}{x^4\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda}=\sum_{i=1}^{n}\frac{1-e^{-\frac{\theta}{x^2}}}{1+\lambda-\lambda e^{-\frac{\theta}{x^2}}},$$

$$\frac{\partial^2 l}{\partial\lambda\partial\theta}=\sum_{i=1}^{n}\frac{2e^{-\frac{\theta}{x^2}}}{x^2\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^2}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\frac{\theta}{x^2}}}{x^2\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\theta}=-\sum_{i=1}^{n}\frac{1}{x^2}+\sum_{i=1}^{n}\frac{\lambda e^{-\frac{\theta}{x^2}}}{x^2\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)}.$$

6. Bayesian Estimation based on non-informative prior

In this Section, the Bayes estimators are derived using a uniform prior.

The joint uniform prior for α , λ and θ is

$$p(\alpha,\lambda,\theta/x)\propto 1 \qquad (10)$$

The joint Posterior distribution is

$$p(\alpha,\lambda,\theta|\underline{x})=\frac{L(\alpha,\lambda,\theta/\underline{x})\,p(\alpha,\lambda,\theta/\underline{x})}{\int_0^\infty\int_{-1}^1\int_0^\infty L(\alpha,\lambda,\theta/\underline{x})\,p(\alpha,\lambda,\theta/\underline{x})d\alpha d\lambda d\theta} \qquad (11)$$

Substituting equation (19) and (22) in above equations, we have,

$$p(\alpha,\lambda,\theta/\underline{x})=\frac{\alpha^n\prod_{i=1}^{n}\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}\left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^2}}\right]\left[e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}\left[1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}}{\int_0^\infty\int_{-1}^1\int_0^\infty\alpha^n\prod_{i=1}^{n}\frac{2\theta}{x^3}e^{-\frac{\theta}{x^2}}\left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^2}}\right]\left[e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}\left[1+\lambda-\lambda\, e^{-\frac{\theta}{x^2}}\right]^{\alpha-1}d\alpha d\lambda d\theta}$$

The Bayes estimators are obtained under loss function, that are;

$$E[u(\alpha,\lambda,\theta)] = \frac{\iiint u(\alpha,\lambda,\theta)\alpha^n \prod_{i=1}^{n} \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}} \left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^2}}\right] \left[e^{-\frac{\theta}{x^2}}\right]^{\alpha-1} \left[1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right]^{\alpha-1} d\alpha d\lambda d\theta}{\iiint \alpha^n \prod_{i=1}^{n} \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}} \left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^2}}\right] \left[e^{-\frac{\theta}{x^2}}\right]^{\alpha-1} \left[1+\lambda-\lambda\, e^{-\frac{\theta}{x^2}}\right]^{\alpha-1} d\alpha d\lambda d\theta}$$

It may be noted integration in the denominator which becomes intractable and quite laborious to solve. Therefore, use Lindley's approximation proposed by (Lindley, 1980), which solve the ratio of integrals as a whole and produces a single numerical result. According to Lindley approximation;

$$E[u(\alpha,\lambda,\theta)] = \frac{\iiint u(\alpha,\lambda,\theta) e^{\ln(\alpha,\lambda,\theta)+\rho(\alpha,\lambda,\theta)}\, d\alpha d\lambda d\theta}{\iiint e^{\ln(\alpha,\lambda,\theta)+\rho(\alpha,\lambda,\theta)}\, d\alpha d\lambda d\theta} \tag{12}$$

where, $u(\alpha,\lambda,\theta)$ is the function of α,λ and θ, $ln(\alpha,\lambda,\theta)$ is log of likelihood, $\rho(\alpha,\lambda,\theta)$ is log joint prior of α,λ and θ. The expression (12) is approximated as;

$$E[u(\alpha,\lambda,\theta)] = u(\hat{\alpha},\hat{\lambda},\hat{\theta}) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})] \tag{13}$$

where $\hat{\alpha},\hat{\lambda}$ and $\hat{\theta}$ are the MLE's of α,λ and θ respectively.

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \qquad i = 1,2,3,$$

and,

$$a_4 = u_1 2\sigma_{12} + u_1 3\sigma_{13} + u_2 3\sigma_{23},$$

$$a_5 = \frac{1}{2}(u_1 1\sigma_{11} + u_2\sigma_{22} + u_3\sigma_{33}).$$

then,

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}$$

and

$\rho_i = \frac{\partial\rho}{\partial\theta_i}$ $\quad i = 1,2,3$ where $\theta_1 = \alpha$, $\theta_2 = \lambda$, $\theta_3 = \theta$,

$$u_j = \frac{\partial u(\alpha,\lambda,\theta)}{\partial\theta_i},\ u_{ij} = \frac{\partial^2 u(\alpha,\lambda,\theta)}{\partial\theta_i\,\partial\theta_j}, \qquad L_{ij} = \frac{\partial^2 L(\alpha,\lambda,\theta)}{\partial\theta_i\ \partial\theta_j}, \qquad L_{ijk} = \frac{\partial^3 L(\alpha,\lambda,\theta)}{\partial\theta_i\ \partial j\,\partial\theta_k},$$

and σ_{ij} is $(i,j)^{th}$ element of the inverse of the matrix $\{L_{ij}\}$, all evaluated to find the MLE of parameters.

$$\rho = ln(joint\, prior) = 0 \ \text{ so } \rho_1 = \rho_2 = \rho_3 = 0, \text{ and,}$$

$$L_{111} = \frac{2n}{\hat{\alpha}^3}, \qquad L_{121} = L_{211} = 0, \quad L_{113} = L_{311} = 0,$$

$$L_{231} = \sum_{i=1}^{n} \frac{e^{-\frac{\theta}{x^2}}}{x^2\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$L_{221} = -\sum_{i=1}^{n} \frac{\left(1-e^{-\frac{\theta}{x^2}}\right)^2}{\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$L_{331} = -\sum_{i=1}^{n} \frac{\lambda(1+\lambda)e^{-\frac{\theta}{x^2}}}{x^4\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^2},$$

$$L_{232} = -\sum_{i=1}^{n} \frac{4e^{-\frac{\theta}{x^2}}\left(1-2e^{-\frac{\theta}{x^2}}\right)}{x^2\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^3} - (\alpha-1)\sum_{i=1}^{n} \frac{2e^{-\frac{\theta}{x^2}}\left(1-e^{-\frac{\theta}{x^2}}\right)}{x^2\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^3},$$

$$L_{222} = \sum_{i=1}^{n} \frac{2\left(1-2e^{-\frac{\theta}{x^2}}\right)^3}{\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^3} + (\alpha-1)\sum_{i=1}^{n} \frac{2\left(1-e^{-\frac{\theta}{x^2}}\right)^3}{\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^3},$$

$$L_{332} = -\sum_{i=1}^{n} \frac{2e^{-\frac{\theta}{x^2}}\left(1+\lambda+2\lambda e^{-\frac{\theta}{x^2}}\right)}{x^4\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^3} - (\alpha-1)\sum_{i=1}^{n} \frac{e^{-\frac{\theta}{x^2}}\left(1+\lambda+\lambda e^{-\frac{\theta}{x^2}}\right)}{x^4\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^3},$$

$$L_{333} = -\sum_{i=1}^{n} \frac{2}{\theta^3} + \sum_{i=1}^{n} \frac{2\lambda(1+\lambda)e^{-\frac{\theta}{x^2}}\left(1+\lambda+2\lambda e^{-\frac{\theta}{x^2}}\right)}{x^6\left(1+\lambda-2\lambda e^{-\frac{\theta}{x^2}}\right)^3}$$

$$+ (\alpha-1)\sum_{i=1}^{n} \frac{\lambda(1+\lambda)e^{-\frac{\theta}{x^2}}\left(1+\lambda+\lambda e^{-\frac{\theta}{x^2}}\right)}{x^6\left(1+\lambda-\lambda e^{-\frac{\theta}{x^2}}\right)^3}.$$

6.1.1 Bayes Estimators and Posterior risks under SELF

The bayes estimator and posterior risk of α under SELF are defined as,

$$\hat{\alpha}_{SELF} = E\left[\hat{\alpha}/\underline{x}\right] = \hat{\alpha} + \tfrac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}), \tag{14}$$

and posterior risk or variance are given by,

$$Risk\ (\hat{\alpha}_{SELF}) = Var\left[\hat{\alpha}|\underline{x}\right] = E\left(\hat{\alpha}^2|\underline{x}\right) - \left[E\left(\hat{\alpha}|\underline{x}\right)^2\right],$$

$$Risk\ (\hat{\alpha}_{SELF}) = \sigma_{11} - \left[\tfrac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})\right]. \tag{15}$$

The Bayes estimator and posterior risk of λ are,

$$\hat{\lambda}_{SELF} = E\left[\hat{\lambda}/\underline{x}\right] = \hat{\lambda} + \tfrac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}), \tag{16}$$

$$Risk\left(\hat{\lambda}_{SELF}\right)=Var\left[\hat{\lambda}/\underline{x}\right]=E\left(\hat{\lambda}^{2}/\underline{x}\right)-\left[E\left(\frac{\hat{\lambda}}{\underline{x}}\right)^{2}\right],$$

$$Risk\left(\hat{\lambda}_{SELF}\right)=\sigma_{22}-\left[\frac{1}{2}(A\sigma_{12}+B\sigma_{22}+C\sigma_{32})\right]^{2}. \quad (17)$$

The Bayes estimator and posterior risk of θ are,

$$\hat{\theta}_{SELF}=E\left[\hat{\theta}|\underline{x}\right]=\hat{\theta}+\frac{1}{2}(A\sigma_{13}+B\sigma_{23}+C\sigma_{33}), \quad (18)$$

$$Risk\left(\hat{\theta}_{SELF}\right)=\sigma_{33}-\left[\frac{1}{2}(A\sigma_{13}+B\sigma_{23}+C\sigma_{33})\right]^{2}. \quad (19)$$

7. Bayesian Estimation using Informative Prior

In this Section we will use the exponential and uniform prior as an informative prior.

7.1 Posterior distribution by using exponential uniform prior

Exponential prior is the most important prior in Bayesian analysis that is much popular in literature. We take exponential prior of parameters α , θ, and uniform prior of λ ,which are defined as respectively,

$$\pi(\alpha)\propto e^{-\alpha/a} \qquad a,\alpha>0,$$

$$\pi(\theta)\propto e^{-\theta/b} \qquad b,\theta>0, \text{ and,}$$

$$\pi(\lambda)\propto 1 \qquad -1<\lambda<1.$$

where a, b are hyper-parameters which are considered as fixed quantities and these can be determined by elicitation of predictive prior distribution.

Consider the joint prior distribution as,

$$p(\alpha,\lambda,\theta)=e^{-\left(\frac{\alpha}{a}+\frac{\theta}{b}\right)}. \quad (20)$$

So, by using (8) and (20), joint posterior distribution is defined as,

$$p(\alpha,\lambda,\theta|\underline{x})=\frac{\alpha^{n}\prod_{i=1}^{n}\frac{2\theta}{x^{3}}e^{-\frac{\theta}{x^{2}}}\left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^{2}}}\right]\left[e^{-\frac{\theta}{x^{2}}}\right]^{\alpha-1}\left[1+\lambda-\lambda\, e^{-\frac{\theta}{x^{2}}}\right]^{\alpha-1}\times e^{-\left(\frac{\alpha}{a}+\frac{\theta}{b}\right)}}{\int_{0}^{\infty}\int_{-1}^{1}\int_{0}^{\infty}\alpha^{n}\prod_{i=1}^{n}\frac{2\theta}{x^{3}}e^{-\frac{\theta}{x^{2}}}\left[1+\lambda-2\lambda\, e^{-\frac{\theta}{x^{2}}}\right]\left[e^{-\frac{\theta}{x^{2}}}\right]^{\alpha-1}\left[1+\lambda-\lambda\, e^{-\frac{\theta}{x^{2}}}\right]^{\alpha-1}\times e^{-\left(\frac{\alpha}{a}+\frac{\theta}{b}\right)}d_{\alpha}d_{\lambda}d_{\theta}}.$$

By using Lindley's approximation, to solve the ratio of integrals as a whole and produces a single numerical result. The Lindley's approximation helps us to obtain the Bayes estimator and risk under SELF.

7.2 Bayes Estimator and posterior risk using SELF

The Bayes estimator for α is given as,

$$\hat{\alpha}_{SELF}=E\left[\hat{\alpha}|\underline{x}\right]=\hat{\alpha}+\left(-\frac{1}{a}\sigma_{11}-\frac{1}{b}\sigma_{13}\right)\frac{1}{2}(A\sigma_{11}+B\sigma_{21}+C\sigma_{31}), \quad (21)$$

and posterior risk or variance for α is,

$$Risk\left(\hat{\alpha}_{SELF}\right)=Var\left[\hat{\alpha}|\underline{x}\right]=E\left(\hat{\alpha}^{2}|\underline{x}\right)-\left[E(\hat{\alpha}|\underline{x})\right]^{2},$$

$$Risk\ (\hat{\alpha}_{SELF}) = \sigma_{11} - \left(-\frac{1}{a}\sigma_{11} - \frac{1}{b}\sigma_{13}\right)^2 - \left[\frac{1}{2}(A\sigma_{11} + B\sigma_{21} + C\sigma_{31})\right]^2 - \left(-\frac{1}{a}\sigma_{11} - \frac{1}{b}\sigma_{13}\right)(A\sigma_{11} + B\sigma_{21} + C\sigma_{31}). \quad (22)$$

The Bayes estimator for λ is,

$$\hat{\lambda}_{SELF} = E[\hat{\lambda}|\underline{x}] = \hat{\lambda} + \left(-\frac{1}{a}\sigma_{21} - \frac{1}{b}\sigma_{23}\right) + \frac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}), \quad (23)$$

and posterior risk or variance is,

$$Risk\ (\hat{\lambda}_{SELF}) = \sigma_{22} - \left(-\frac{1}{a}\sigma_{21} - \frac{1}{b}\sigma_{23}\right)^2 - \left[\frac{1}{2}(A\sigma_{12} + B\sigma_{22} + C\sigma_{32})\right]^2 - \left(-\frac{1}{a}\sigma_{21} - \frac{1}{b}\sigma_{23}\right)(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}). \quad (24)$$

The Bayes estimator for θ is,

$$\hat{\theta}_{SELF} = E[\hat{\theta}|\underline{x}] = \hat{\theta} + \left(-\frac{1}{a}\sigma_{13} - \frac{1}{b}\sigma_{33}\right) + \frac{1}{2}(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}), \quad (25)$$

and posterior risk or variance for θ is,

$$Risk\ (\hat{\theta}_{SELF}) = \sigma_{33} - \left(-\frac{1}{a}\sigma_{13} - \frac{1}{b}\sigma_{33}\right) - \left[\frac{1}{2}(A\sigma_{13} + B\sigma_{23} + C\sigma_{33})\right]^2 - \left(-\frac{1}{a}\sigma_{13} - \frac{1}{b}\sigma_{33}\right)(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}). \quad (26)$$

8. Monte Carlo Simulation Study

In this Section, to check the performance of estimator by using Monte Carlo Simulation scheme. Using the programming routine in R-language, taking 1000 replications for different sample sizes, n = 80, 100, 150, 200, 250, 300, 400, 500 and 1000. Bayes estimators MLEs obtained from theoretical are estimated here.

To find the Bays estimator and its risk, the hyper parameters of the prior distribution

- Bayes Estimates and MLE of α and their risks in parenthesis using Uniform and Informative Priors under SELF when $\alpha = 1$, and take different values of hyper parameter for informative prior $a = 4\ and\ 0.4\ b = 2\ and\ 1.5$
- Bayes estimates and MLE of θ and their risks in parenthesis using uniform and informative Priors under SELF when $\theta = 2.4$ using informative the value of hyper parameters are $a = 4\ and\ 0.4$, $b = 2\ and\ 1.5$
- Similarly, set transmuted parameter $\lambda = -0.7$ and different values of hyper parameter as $a = 4\ and\ 0.4\ b = 2\ and\ 1.5$
- Numerical results for simulation study are presented in Tables.
- Similarly, α_2, λ_2 , $\theta_2$represent again informative prior in which the hyper parameter are given by e_2 and f_2.

Table 1: Bayes Estimates and MLE of α (risks) using uniform prior when $\alpha = 1$

n	MLE	Uniform Prior	Informative prior	
		α_{SELF}	$\alpha_{1\ SELF}$	$\alpha_{2\ SELF}$
80	0.09337 (0.00012)	0.09558 (0.11171)	0.09652 (0.00210)	0.09548 (0.01375)
100	0.09400 (0.06466)	0.09560 (0.06575)	0.09607 (0.00010)	0.09696 (0.00020)
150	0.09259 (0.04556)	0.09534 (0.01334)	0.09616 (0.01403)	0.09708 (0.02021)
200	0.09328 (0.03431)	0.09525 (0.02920)	0.09585 (0.02616)	0.09578 (0.02914)
250	0.09292 (0.02731)	0.09449 (0.02535)	0.09496 (0.02410)	0.09519 (0.02361)
300	0.09382 (0.02179)	0.09464 (0.02132)	0.09510 (0.02063)	0.09517 (0.02047)
400	0.09469 (0.01664)	0.09530 (0.01682)	0.09566 (0.01598)	0.09577 (0.01590)
500	0.09402 (0.01312)	0.09451 (0.01295)	0.09479 (0.01271)	0.0948 (0.01266)
1000	0.09423 (0.02418)	0.09447 (0.02403)	0.09461 (0.02381)	0.09461 (0.02376)

Table 2: Bayes Estimates and MLE of λ using uniform and Informative prior when $\lambda = -0.7$

n	MLE	Uniform Prior	Informative Prior	
		λ_{SELF}	$\lambda_{1\ SELF}$	$\lambda_{2\ SELF}$
80	0.99346 (0.03428)	0.99985 (0.00030)	0.99999 (0.00032)	1.00240 (0.00103)
100	0.99297 (0.03215)	0.99926 (0.00013)	0.99943 (0.00013)	0.99929 (0.00013)
150	0.99108 (0.02708)	0.99460 (0.04144)	0.99465 (0.04221)	0.99547 (0.04871)
200	0.98970 (0.02538)	0.992342 (0.03134)	0.99236 (0.03153)	0.99312 (0.03819)
250	0.98907 (0.02287)	0.99129 (0.02690)	0.99131 (0.02702)	0.99206 (0.03145)
300	0.98709 (0.02655)	0.98947 (0.03092)	0.98958 (0.03132)	0.99003 (0.02998)
400	0.98582 (0.02293)	0.98768 (0.02555)	0.98775 (0.02576)	0.98725 (0.02738)
500	0.98585 (0.01846)	0.98731 (0.02009)	0.98737 (0.02022)	0.98740 (0.02010)
1000	0.98052 (0.01469)	0.98134 (0.01519)	0.98136 (0.01521)	0.98173 (0.01493)

Table 3: Bayes Estimates and MLE of θ using uniform and informative prior when $\boldsymbol{\theta} = \mathbf{2.4}$

n	MLE	Uniform Prior	Informative Prior	
		$\boldsymbol{\theta_{SELF}}$	$\boldsymbol{\theta_{1\ SELF}}$	$\boldsymbol{\theta_{2\ SELF}}$
80	9.1136 (1.19533)	8.36357 (43.83017)	8.06674 (53.50167)	7.13047 (43.37427)
100	7.0545 (0.54209)	6.72592 (9.52465)	6.59146 (9.66930)	7.071876 (18.11027)
150	9.996 (0.96895)	9.26425 (1.37416)	9.02369 (1.80482)	7.26754 (0.68211)
200	9.9965 (0.72025)	9.48866 (0.16780)	9.30983 (0.12087)	7.45586 (0.07722)
250	9.9908 (0.57510)	9.58474 (0.38461)	9.44196 (0.23989)	7.56136 (0.11201)
300	7.9895 (0.26395)	7.80550 (0.22781)	7.67376 (0.16074)	7.63158 (0.13110)
400	7.9882 (0.19844)	7.85126 (0.17850)	7.75223 (0.14101)	7.72114 (0.12479)
500	7.9905 (0.15850)	7.88192 (0.14619)	7.80281 (0.12244)	7.77623 (0.11235)
1000	7.9894 (0.08086)	7.93640 (0.07794)	7.89605 (0.07196)	7.88286 (0.06935)

- In table 1, when $\alpha = 1$ the sample size is $80, 100, 150$ risk under SELF for Bayesian estimators are greater than the MLE. As increases sample size $n = 200$ risk become smaller than the MLE.
- Also this Table shows the Informative prior under SELF calculate two times.$\alpha_{1\,SELF}$ for informative prior when hyper parameter values $e_1 = 4$ and $f_1 = 2$. When α_{2SELF} hyper parameter values used $e_2 = 0.4$ and $f_1 = 1.5$. This show that the hyper parameter values decreases the results of risk function better than the risks of non informative prior.
- In table 2 the value of $\lambda = -0.7$ when the $n = 80, 100$ the risk of Bayes estimator is less than the MLE but the sample size increase $n = 150,\ 200, 250,\ 300,\ 350, 400, 500$ and 1000, show the risks of MLE is less than the Bayesian estimator using informative and non-informative prior under SELF. Hence, for transmuted parameter λ MLE is the best estimator for ETIR distribution.
- Bayes estimates and MLE of $\theta = 2.4$. This table show when $n = 80, 100, 150$ risk values of MLE is minimum the Uniform prior. when sample size increase then the risk of Bayes estimator are better results as compare to MLE.

Hence we conclude that when $n < 200$ MLE is the better estimates as compare to Bayesian estimates. When $n > 200$ Bayes estimators produce the good results than the MLE's.

9 Real Life Application

In this section, we used different real life data sets were used to compare the performance of the MLE's and Bayes Estimators of ETIRD of uniform prior under SELF.

9.1 Data Set 1: Coating Weigh of Iron Sheet

This data consists of 72 observations on coating weigh of Iron sheets by chemical method on top center side (TCS). Previously this data set is used by (Rao & Mbwambo.,2019)

The data set is given below.

36.8, 47.2, 35.6, 36.7, 55.8, 58.7, 42.3, 37.8, 55.4, 45.2, 31.8, 48.3, 45.3, 48.5, 52.8, 45.4, 49.8, 48.2, 54.5, 50.1, 48.4, 44.2, 41.2, 47.2, 39.1, 40.7, 40.3, 41.2, 30.4, 42.8, 38.9, 34.0, 33.2, 56.8, 52.6, 40.5, 40.6, 45.8, 58.9, 28.7, 37.3, 36.8, 40.2, 58.2, 59.2, 42.8, 46.3, 61.2, 58.4, 38.5, 34.2, 41.3, 42.6, 43.1, 42.3, 54.2, 44.9, 42.8, 47.1, 38.9, 42.8, 29.4, 32.7, 40.1, 33.2, 31.6, 36.2, 33.6, 32.9, 34.5, 33.7, 39.9.

The results for above data set are given in Table 4.

Table 4: Bayes Estimates & MLE of α, λ and θ and their risks in parenthesis using Uniform prior

	MLE	Uniform Prior $SELF_{NI}$	Informative Prior $SELF_I$
α	0.06340 (0.03809)	0.06564 (0.03473)	0.03735 (0.00062)
λ	0.99952 (0.00749)	0.99729 (0.00129)	0.99767 (0.00062)
θ	7.319533 (0.00747)	7.319535 (0.00724)	7.322534 (0.00510)

- For α, by using uniform prior and informative prior, show that the Bayes estimates consist the small risk of the MLE.
- For λ, show that the Bayes estimator is better for MLE.
- Similarly for θ by using uniform prior and informative prior show that the small risks than MLE.

9.2 Data Set 2: Aircraft Wind Speed

This data set is the failure times of 84 Aircraft wind speed. These failure times data is used by (Khan, 2018) for the special case of Transmuted Generalized Inverse Rayleigh (TGIR) distribution.

The data set is given below:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 5: Bayes Estimates and MLE of α, λ and θ and their risks in parenthesis using Uniform prior.

	MLE	Uniform Prior $SELF_{NI}$	Informative Prior $SELF_{I}$
α	0.10203 (0.00012)	0.10409 (0.00011)	0.09792 (0.00010)
λ	0.99528 (0.05283)	0.00022 (0.00022)	0.97823 (0.00011)
θ	1.17124 (0.00572)	1.17124 (0.00371)	1.17061 (0.00215)

Bayes estimators are obtained under SELF using uniform and informative priors are attained using the above real life data set to compare with MLE.

- In table 5, we compute the scale parameter θ, transmuted parameter λ and shape parameter α, by using the uniform and informative priors.
- Bayes estimates for SELF contained the small risks than MLE. So Bayes estimators are better as compare to MLE.

9.3 Data Set 3: Treatment of Chemotherapy

This data set consists of 46 patients of Survival times (in year) to given treatment of Chemotherapy. This data is used by (Bekker et al., 2000) and (Murthy et al., 2004). The data is given below.

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203,0.260, 0.282, 0.296,0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540,0.570, 0.641, 0.644, 0.696, 0.841, 0.863,1.099,1.219,1.271,1.326,1.447,1.485,1.553,1.581,1.589,2.178,2.343,2.416,2.444,2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Table 6: Bayes Estimates and MLE of α, λ and θ and their risks in parenthesis using Uniform prior.

	MLE	Uniform Prior $SELF_{NI}$	Informative Prior $SELF_{I}$
α	0.02926 (0.01289)	0.03187 (0.00827)	0.02996 (0.00256)

$\boldsymbol{\lambda}$	0.99993 (0.00159)	0.99900 (0.00026)	0.99902 (0.00019)
$\boldsymbol{\theta}$	1.61739 (0.01239)	1.61736 (0.01129)	1.61760 (0.000151)

- It can be observed that table 6, show the Bayes Estimator and MLE using the uniform and Informative Prior.
- Parameter α the risk of $\boldsymbol{SELF_{NI}}$, $\boldsymbol{SELF_{I}}$ are less than the MLE. So SELF is better than MLE.
- Parameter λ also show the Bayesian estimators are better performed than MLE because Bayes estimators consist of the small risks.
- Parameter θ show the risk function of uniform and informative prior both less than the MLE risk. son in this data set Bayes estimators indicate the best result compared to MLE.

9.4 Data Set 4: Fatigue Lifetime

The data set is given by (Birnbaum & Saunders, 1969) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data also presented (Shanker et al., 2015).

The data set are given below:

5 , 25 ,31 ,32 ,34 ,35, 38 ,39, 39, 40 ,42 ,43, 43, 43 ,44, 44, 47, 47 ,48, 59, 59, 63 ,63, 64, 64, 65, 65, 65, 66, 66 ,66 ,66 ,66 ,67, 67, 67, 68, 69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76, 76, 77, 77, 77, 77, 77, 77, 79, 79, 80 ,81, 83, 83, 84 ,86, 86, 87 ,90, 91 ,92 ,92, 92 ,92 ,93 ,94 ,97, 98, 98 ,99, 101, 103, 105, 109, 136, 147

Table 7: Bayes Estimates and MLE of α, λ and θ and their risks in parenthesis using Uniform prior.

	MLE	Uniform Prior $\boldsymbol{SELF_{NI}}$	Informative Prior $\boldsymbol{SELF_{I}}$
$\boldsymbol{\alpha}$	0.02496 (0.01529)	0.02520 (0.01514)	0.02517 (0.01401)
$\boldsymbol{\lambda}$	5.17020 (0.05212)	5.14683 (0.05158)	5.16039 (0.0203)
$\boldsymbol{\theta}$	0.06120 (0.02143)	0.06290 (0.01948)	0.06282 (0.01966)

- It can be observed that, table 7, show the Bayes Estimator and MLE using the uniform Prior and informative prior.

- Bayes estimates under SELF using uniform prior and informative prior indicates the small value of risk than Maximum likelihood estimates for α , λ and θ also have the small risks for the same loss function.

10 Conclusion

In this study, we derived a new distribution named as Exponentiated Transmuted Inverse Rayleigh Distribution (ETIRD consisting of three parameters α, λ and θ. The mathematical properties of ETIRD are derived in detail. Furthermore, the parameters of the distribution are derived using the ML Estimation technique and Bayesian estimation method. The Bayes estimators and their risks are obtained using an approximation technique known as Lindley's approximation method. The Bayes estimator under SELF using non-informative and informative Priors (uniform prior) are evaluated. We use the simulation study to check the performance of estimators. It observed that the Bayes estimators yield good results for large sample sizes than the MLE. We can say that our study follows the large sample property of statistical inference. For ETIRD Bayes estimators yields good results for small and large sample sizes than MLE. However, for large sample sizes Bayes estimators indicate the better results than small sample sizes under SELF. By taking the different true values of parameters, it can be notice that posterior risks of Bayes estimators are small under SELF. The performance of Bayes estimator in real life application is checked using four real life data sets and then compared the numerical results of MLE with Bayes estimators under SELF. Although we are using an approximation, technique to obtain the Bayes estimators under SELF but their performance is better than the most commonly method used in classical statistics maximum likelihood method.

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